

# Anti-Wick and Weyl quantization on ultradistribution spaces

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## Abstract

The connection between the Anti-Wick and Weyl quantization is given for certain class of global symbols, which corresponding pseudodifferential operators act continuously on the space of tempered ultradistributions of Beurling, respectively, of Roumieu type. The largest subspace of ultradistributions is found for which the convolution with the gaussian kernel exist. This gives a way to extend the definition of Anti-Wick quantization for symbols that are not necessarily tempered ultradistributions.

## 0 Introduction

The Anti-Wick and the Weyl quantization of global symbols, as well as their connection, in the case of Schwartz distributions was vastly studied during the years (see for example [10] and [19] for a systematic approach to the theory). The importance in studying the Anti-Wick quantization lies in the facts that real valued symbols give rise to formally self-adjoint operators and positive symbols give rise to positive operators. On the other hand the Weyl quantization is important because it is closely connected with the Wigner transform and also, the Weyl quantization of real valued symbol is formally self-adjoint operator.

The results that we give here are related to the global symbol classes defined and studied in [16], which corresponding operators act continuously on the space of tempered ultradistributions of Beurling, resp. Roumieu type.

For a symbol  $a$  which is an element of the space of tempered (ultra)distributions, its Anti-Wick quantization is equal to the Weyl quantization of a symbol  $b$  that is given as the convolution of  $a$  and the gaussian kernel  $e^{-|\cdot|^2}$ . The purpose of this paper is twofold. In the first part we extend results from [10] (see also [19]) to ultradistributions. More precisely, we give the connection of Anti-Wick and Weyl quantization for symbols belonging to specific symbol classes developed by one of the authors in [16]. The last two sections are devoted to finding the largest subspace of ultradistributions for which the convolution with the gaussian kernel exist. The answer to this question in the case of Schwartz distributions was already given in [21]. This gives a way to extend the definition of Anti-Wick operators with symbols that are not necessarily tempered ultradistributions. In particular, we prove

theorem 5.1, which gives such class of symbols.

The paper is organized as follows:

**Section 1** contains some basic facts concerning spaces of ultradistribution.

In **Section 2** we recall important results related to the symbol classes and their corresponding pseudodifferential operators defined and studied in [16].

**Section 3** is devoted to the connection between the Anti-Wick and Weyl quantization of symbols belonging to the mentioned symbol classes.

In **Section 4** we find the largest subspace of ultradistributions for which the convolution with  $e^{s|\cdot|^2}$ ,  $s \in \mathbb{R} \setminus \{0\}$ , exist.

In **Section 5** we extend the definition of Anti-Wick operators for symbols that are not necessarily tempered ultradistributions, by using the results obtained in the previous sections.

## 1 Preliminaries

The sets of natural, integer, positive integer, real and complex numbers are denoted by  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Z}_+$ ,  $\mathbb{R}$ ,  $\mathbb{C}$ . We use the symbols for  $x \in \mathbb{R}^d$ :  $\langle x \rangle = (1 + |x|^2)^{1/2}$ ,  $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_d}$ ,  $D_j^{\alpha_j} = i^{-1} \partial^{\alpha_j} / \partial x^{\alpha_j}$ ,  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}^d$ . If  $z \in \mathbb{C}^d$ , by  $z^2$  we will denote  $z_1^2 + \dots + z_d^2$ . Note that, if  $x \in \mathbb{R}^d$ ,  $x^2 = |x|^2$ .

Following [6], we denote by  $M_p$  a sequence of positive numbers  $M_0 = 1$  so that:

$$(M.1) \quad M_p^2 \leq M_{p-1} M_{p+1}, \quad p \in \mathbb{Z}_+;$$

$$(M.2) \quad M_p \leq c_0 H^p \min_{0 \leq q \leq p} \{M_{p-q} M_q\}, \quad p, q \in \mathbb{N}, \text{ for some } c_0, H \geq 1;$$

$$(M.3) \quad \sum_{p=q+1}^{\infty} \frac{M_{p-1}}{M_p} \leq c_0 q \frac{M_q}{M_{q+1}}, \quad q \in \mathbb{Z}_+,$$

although in some assertions we could assume the weaker ones  $(M.2)'$  and  $(M.3)'$  (see [6]). For a multi-index  $\alpha \in \mathbb{N}^d$ ,  $M_\alpha$  will mean  $M_{|\alpha|}$ ,  $|\alpha| = \alpha_1 + \dots + \alpha_d$ . Recall,  $m_p = M_p / M_{p-1}$ ,  $p \in \mathbb{Z}_+$  and the associated function for the sequence  $M_p$  is defined by

$$M(\rho) = \sup_{p \in \mathbb{N}} \log_+ \frac{\rho^p}{M_p}, \quad \rho > 0.$$

It is non-negative, continuous, monotonically increasing function, which vanishes for sufficiently small  $\rho > 0$  and increases more rapidly than  $(\ln \rho)^p$  when  $\rho$  tends to infinity, for any  $p \in \mathbb{N}$ .

Let  $U \subseteq \mathbb{R}^d$  be an open set and  $K \subset\subset U$  (we will use always this notation for a compact subset of an open set). Then  $\mathcal{E}^{\{M_p\}, h}(K)$  is the space of all  $\varphi \in \mathcal{C}^\infty(U)$  which satisfy  $\sup_{\alpha \in \mathbb{N}^d} \sup_{x \in K} \frac{|D^\alpha \varphi(x)|}{h^\alpha M_\alpha} < \infty$  and  $\mathcal{D}_K^{\{M_p\}, h}$  is the space of all  $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$  with supports

in  $K$ , which satisfy  $\sup_{\alpha \in \mathbb{N}^d} \sup_{x \in K} \frac{|D^\alpha \varphi(x)|}{h^\alpha M_\alpha} < \infty$ ;

$$\mathcal{E}^{(M_p)}(U) = \varprojlim_{K \subset\subset U} \varprojlim_{h \rightarrow 0} \mathcal{E}^{\{M_p\}, h}(K), \quad \mathcal{E}^{\{M_p\}}(U) = \varprojlim_{K \subset\subset U} \varinjlim_{h \rightarrow \infty} \mathcal{E}^{\{M_p\}, h}(K),$$

$$\mathcal{D}^{(M_p)}(U) = \lim_{K \subset \subset U} \lim_{h \rightarrow 0} \mathcal{D}_K^{\{M_p\}, h}, \quad \mathcal{D}^{\{M_p\}}(U) = \lim_{K \subset \subset U} \lim_{h \rightarrow \infty} \mathcal{D}_K^{\{M_p\}, h}.$$

The spaces of ultradistributions and ultradistributions with compact support of Beurling and Roumieu type are defined as the strong duals of  $\mathcal{D}^{(M_p)}(U)$  and  $\mathcal{E}^{(M_p)}(U)$ , resp.  $\mathcal{D}^{\{M_p\}}(U)$  and  $\mathcal{E}^{\{M_p\}}(U)$ . For the properties of these spaces, we refer to [6], [7] and [8]. In the future we will not emphasize the set  $U$  when  $U = \mathbb{R}^d$ . Also, the common notation for the symbols  $(M_p)$  and  $\{M_p\}$  will be  $*$ .

For  $f \in L^1$ , its Fourier transform is defined by  $(\mathcal{F}f)(\xi) = \int_{\mathbb{R}^d} e^{-ix\xi} f(x) dx$ ,  $\xi \in \mathbb{R}^d$ .

By  $\mathfrak{R}$  is denoted a set of positive sequences which monotonically increases to infinity. For  $(r_p) \in \mathfrak{R}$ , consider the sequence  $N_0 = 1$ ,  $N_p = M_p \prod_{j=1}^p r_j$ ,  $p \in \mathbb{Z}_+$ . One easily sees that this sequence satisfies (M.1) and (M.3)' and its associated function will be denoted by  $N_{r_p}(\rho)$ , i.e.  $N_{r_p}(\rho) = \sup_{p \in \mathbb{N}} \log_+ \frac{\rho^p}{M_p \prod_{j=1}^p r_j}$ ,  $\rho > 0$ . Note, for given  $(r_p)$  and every  $k > 0$  there is  $\rho_0 > 0$  such that  $N_{r_p}(\rho) \leq M(k\rho)$ , for  $\rho > \rho_0$ . In [8] it is proven that for each  $K \subset \subset \mathbb{R}^d$ , the topology of  $\mathcal{D}_K^{\{M_p\}} = \varinjlim_{h \rightarrow \infty} \mathcal{D}_K^{\{M_p\}, h}$  is generated by the seminorms

$$p_{(t_j), K}(\varphi) = \sup_{\alpha \in \mathbb{N}^d} \frac{\|D^\alpha \varphi\|_{L^\infty}}{M_\alpha \prod_{j=1}^{|\alpha|} t_j}, \text{ where } (t_j) \in \mathfrak{R}. \text{ In [15] the following lemma is proven.}$$

**Lemma 1.1.** *Let  $(k_p) \in \mathfrak{R}$ . There exists  $(k'_p) \in \mathfrak{R}$  such that  $k'_p \leq k_p$  and*

$$\prod_{j=1}^{p+q} k'_j \leq 2^{p+q} \prod_{j=1}^p k'_j \cdot \prod_{j=1}^q k'_j, \text{ for all } p, q \in \mathbb{Z}_+.$$

Hence, for every  $(k_p) \in \mathfrak{R}$ , we can find  $(k'_p) \in \mathfrak{R}$ , as in lemma 1.1, such that  $N_{k_p}(\rho) \leq N_{k'_p}(\rho)$ ,  $\rho > 0$  and the sequence  $N_0 = 1$ ,  $N_p = M_p \prod_{j=1}^p k'_j$ ,  $p \in \mathbb{Z}_+$ , satisfies (M.2) if  $M_p$  does.

From now on, we always assume that  $M_p$  satisfies (M.1), (M.2) and (M.3). It is said that  $P(\xi) = \sum_{\alpha \in \mathbb{N}^d} c_\alpha \xi^\alpha$ ,  $\xi \in \mathbb{R}^d$ , is an ultrapolynomial of the class  $(M_p)$ , resp.  $\{M_p\}$ , whenever the coefficients  $c_\alpha$  satisfy the estimate  $|c_\alpha| \leq CL^{|\alpha|}/M_\alpha$ ,  $\alpha \in \mathbb{N}^d$  for some  $L > 0$  and  $C > 0$ , resp. for every  $L > 0$  and some  $C_L > 0$ . The corresponding operator  $P(D) = \sum_{\alpha} c_\alpha D^\alpha$  is an ultradifferential operator of the class  $(M_p)$ , resp.  $\{M_p\}$  and they act continuously on  $\mathcal{E}^{(M_p)}(U)$  and  $\mathcal{D}^{(M_p)}(U)$ , resp.  $\mathcal{E}^{\{M_p\}}(U)$  and  $\mathcal{D}^{\{M_p\}}(U)$  and the corresponding spaces of ultradistributions. In [15] a special class of ultrapolynomials of class  $*$  were constructed. We summarize the results obtained there in the following proposition.

**Proposition 1.1.** *Let  $c > 0$  and  $k > 0$ , resp.  $c > 0$  and  $(k_p) \in \mathfrak{R}$  are arbitrary but fixed. Then there exist  $l > 0$  and  $q \in \mathbb{Z}_+$ , resp. there exist  $(l_p) \in \mathfrak{R}$  and  $q \in \mathbb{Z}_+$  such that*

$$P_l(z) = \prod_{j=q}^{\infty} \left(1 + \frac{z^2}{l^2 m_j^2}\right), \text{ resp. } P_{l_p}(z) = \prod_{j=q}^{\infty} \left(1 + \frac{z^2}{l_p^2 m_j^2}\right), \text{ is an entire function that doesn't}$$

*have zeroes on the strip  $W = \mathbb{R}^d + i\{y \in \mathbb{R}^d \mid |y_j| \leq c, j = 1, \dots, d\}$ .  $P_l(x)$ , resp.  $P_{l_p}(x)$ , is an*

ultrapolynomial of class  $*$ . Moreover  $|P_l(z)| \geq \tilde{C}e^{M(|z|/k)}$ , resp.  $|P_{l_p}(z)| \geq \tilde{C}e^{N_{k_p}(|z|)}$ ,  $z \in W$ , for some  $\tilde{C} > 0$  and  $\left| \partial_x^\alpha \frac{1}{P_l(x)} \right| \leq C \cdot \frac{\alpha!}{r^{|\alpha|}} e^{-M(|x|/k)}$ , resp.  $\left| \partial_x^\alpha \frac{1}{P_{l_p}(x)} \right| \leq C \cdot \frac{\alpha!}{r^{|\alpha|}} e^{-N_{k_p}(|x|)}$ ,  $x \in \mathbb{R}^d$ ,  $\alpha \in \mathbb{N}^d$ , where  $C$  depends on  $k$  and  $l$ , resp.  $(k_p)$  and  $(l_p)$ , and  $M_p$ ;  $r \leq c$  arbitrary but fixed.

We denote by  $\mathcal{S}_2^{M_p, m}(\mathbb{R}^d)$ ,  $m > 0$ , the space of all smooth functions  $\varphi$  which satisfy

$$\sigma_{m,2}(\varphi) := \left( \sum_{\alpha, \beta \in \mathbb{N}^d} \int_{\mathbb{R}^d} \left| \frac{m^{|\alpha|+|\beta|} \langle x \rangle^{|\alpha|} D^\beta \varphi(x)}{M_\alpha M_\beta} \right|^2 dx \right)^{1/2} < \infty,$$

supplied with the topology induced by the norm  $\sigma_{m,2}$ . The spaces  $\mathcal{S}'^{(M_p)}$  and  $\mathcal{S}'^{\{M_p\}}$  of tempered ultradistributions of Beurling and Roumieu type respectively, are defined as the strong duals of the spaces  $\mathcal{S}^{(M_p)} = \varprojlim_{m \rightarrow \infty} \mathcal{S}_2^{M_p, m}(\mathbb{R}^d)$  and  $\mathcal{S}^{\{M_p\}} = \varinjlim_{m \rightarrow 0} \mathcal{S}_2^{M_p, m}(\mathbb{R}^d)$ , respectively. In [3] (see also [11]) it is proved that the sequence of norms  $\sigma_{m,2}$ ,  $m > 0$ , is equivalent with the sequences of norms  $\|\cdot\|_m$ ,  $m > 0$ , where  $\|\varphi\|_m := \sup_{\alpha \in \mathbb{N}^d} \frac{m^{|\alpha|} \|D^\alpha \varphi(\cdot) e^{M(m|\cdot|)}\|_{L^\infty}}{M_\alpha}$ . If

we denote by  $\mathcal{S}_\infty^{M_p, m}(\mathbb{R}^d)$  the space of all infinitely differentiable functions on  $\mathbb{R}^d$  for which the norm  $\|\cdot\|_m$  is finite (obviously it is a Banach space), then  $\mathcal{S}^{(M_p)}(\mathbb{R}^d) = \varprojlim_{m \rightarrow \infty} \mathcal{S}_\infty^{M_p, m}(\mathbb{R}^d)$  and  $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d) = \varinjlim_{m \rightarrow 0} \mathcal{S}_\infty^{M_p, m}(\mathbb{R}^d)$ . Also, for  $m_2 > m_1$ , the inclusion  $\mathcal{S}_\infty^{M_p, m_2}(\mathbb{R}^d) \rightarrow \mathcal{S}_\infty^{M_p, m_1}(\mathbb{R}^d)$  is a compact mapping. So,  $\mathcal{S}^*(\mathbb{R}^d)$  is a  $(FS)$ -space in  $(M_p)$  case, resp. a  $(DFS)$ -space in the  $\{M_p\}$  case. Moreover, they are nuclear spaces. In [3] (see also [13]) it is proved that  $\mathcal{S}^{\{M_p\}} = \varprojlim_{(r_i), (s_j) \in \mathfrak{R}} \mathcal{S}_{(r_p), (s_q)}^{M_p}$ , where  $\mathcal{S}_{(r_p), (s_q)}^{M_p} = \{\varphi \in \mathcal{C}^\infty(\mathbb{R}^d) \mid \|\varphi\|_{(r_p), (s_q)} < \infty\}$

and  $\|\varphi\|_{(r_p), (s_q)} = \sup_{\alpha \in \mathbb{N}^d} \frac{\|D^\alpha \varphi(x) e^{N_{s_p}(|x|)}\|_{L^\infty}}{M_\alpha \prod_{p=1}^{|\alpha|} r_p}$ . Also, the Fourier transform is a topological automorphism of  $\mathcal{S}^*$  and of  $\mathcal{S}'^*$ .

Denote by  $\mathcal{O}_C^*$  the space of convolutors for  $\mathcal{S}^*$ , i.e. the space of all  $T \in \mathcal{S}'^*$  for which the mapping  $\varphi \mapsto T * \varphi$  is well defined and continuous mapping from  $\mathcal{S}^*$  to itself. Denote by  $\mathcal{O}_M^*$  the space of multipliers for  $\mathcal{S}^*$ , i.e. the space of all  $\psi \in \mathcal{E}^*$  for which the mapping  $\varphi \mapsto \psi \varphi$  is well defined and continuous mapping from  $\mathcal{S}^*$  to itself. For the properties of these spaces we refer to [4].

We need the following kernel theorem for  $\mathcal{S}'^*$  from [16]. The  $(M_p)$  case was already considered in [9] (the authors used the characterization of Fourier-Hermite coefficients of the elements of the space in the proof of the kernel theorem).

**Proposition 1.2.** *The following isomorphisms of locally convex spaces hold*

$$\begin{aligned} \mathcal{S}^*(\mathbb{R}^{d_1}) \hat{\otimes} \mathcal{S}^*(\mathbb{R}^{d_2}) &\cong \mathcal{S}^*(\mathbb{R}^{d_1+d_2}) \cong \mathcal{L}_b(\mathcal{S}'^*(\mathbb{R}^{d_1}), \mathcal{S}^*(\mathbb{R}^{d_2})), \\ \mathcal{S}'^*(\mathbb{R}^{d_1}) \hat{\otimes} \mathcal{S}'^*(\mathbb{R}^{d_2}) &\cong \mathcal{S}'^*(\mathbb{R}^{d_1+d_2}) \cong \mathcal{L}_b(\mathcal{S}^*(\mathbb{R}^{d_1}), \mathcal{S}'^*(\mathbb{R}^{d_2})). \end{aligned}$$

As in [13], we define  $\mathcal{D}_{L^\infty}^*$  by  $\mathcal{D}_{L^\infty}^{(M_p)} = \varprojlim_{h \rightarrow \infty} \mathcal{D}_{L^\infty, h}^{M_p}$ , resp.  $\mathcal{D}_{L^\infty}^{\{M_p\}} = \varinjlim_{h \rightarrow 0} \mathcal{D}_{L^\infty, h}^{M_p}$ , where  $\mathcal{D}_{L^\infty, h}^{M_p}$  is the Banach space of all  $\varphi \in \mathcal{C}^\infty$  for which the norm  $\sup_{\alpha \in \mathbb{N}^d} \frac{h^{|\alpha|} \|D^\alpha \varphi\|_{L^\infty}}{M_\alpha}$  is finite. We define  $\tilde{\mathcal{D}}_{L^\infty}^{\{M_p\}}$  as the space of all  $\mathcal{C}^\infty$  functions such that, for every  $(t_j) \in \mathfrak{R}$ , the norm  $p_{(t_j)}(\varphi) = \sup_{\alpha \in \mathbb{N}^d} \frac{\|D^\alpha \varphi\|_{L^\infty}}{M_\alpha \prod_{j=1}^{|\alpha|} t_j}$  is finite. The space  $\tilde{\mathcal{D}}_{L^\infty}^{\{M_p\}}$  is complete Hausdorff locally convex space because  $\tilde{\mathcal{D}}_{L^\infty}^{\{M_p\}} = \varprojlim_{(t_j) \in \mathfrak{R}} \tilde{\mathcal{D}}_{L^\infty, (t_j)}^{M_p}$ , where  $\tilde{\mathcal{D}}_{L^\infty, (t_j)}^{M_p}$  is the Banach space of all  $\mathcal{C}^\infty$  functions for which the norm  $p_{(t_j)}(\cdot)$  is finite. In [13] it is proved that  $\mathcal{D}_{L^\infty}^{\{M_p\}} = \tilde{\mathcal{D}}_{L^\infty}^{\{M_p\}}$  as sets and the former has a stronger topology than the later. Denote by  $\dot{\mathcal{B}}^{(M_p)}$ , resp.  $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}$  the completion of  $\mathcal{D}^{(M_p)}$ , resp.  $\mathcal{D}^{\{M_p\}}$ , in  $\mathcal{D}_{L^\infty}^{(M_p)}$ , resp.  $\tilde{\mathcal{D}}_{L^\infty}^{\{M_p\}}$ . The strong dual of  $\dot{\mathcal{B}}^{(M_p)}$ , resp.  $\dot{\tilde{\mathcal{B}}}^{\{M_p\}}$ , will be denoted by  $\mathcal{D}'_{L^1}^{(M_p)}$ , resp.  $\tilde{\mathcal{D}}'_{L^1}^{\{M_p\}}$ . For the properties of these spaces we refer to [13].

## 2 A class of pseudo-differential operators

In this section we will give a brief overview of the global symbol classes constructed in [16]. It is important to note that similar symbol classes were considered by M. Cappiello in [1] and [2]. All the results that we give in this section can be found in [16].

Let  $a \in \mathcal{S}'^*(\mathbb{R}^{2d})$ . For  $\tau \in \mathbb{R}$ , consider the ultradistribution

$$K_\tau(x, y) = \mathcal{F}_{\xi \rightarrow x-y}^{-1}(a)((1-\tau)x + \tau y, \xi) \in \mathcal{S}'^*(\mathbb{R}^{2d}). \quad (1)$$

Let  $\text{Op}_\tau(a)$  be the operator from  $\mathcal{S}^*$  to  $\mathcal{S}'^*$  corresponding to the kernel  $K_\tau(x, y)$ , i.e.

$$\langle \text{Op}_\tau(a)u, v \rangle = \langle K_\tau, v \otimes u \rangle, \quad u, v \in \mathcal{S}^*(\mathbb{R}^d). \quad (2)$$

$a$  will be called the  $\tau$ -symbol of the pseudo-differential operator  $\text{Op}_\tau(a)$ . When  $\tau = 0$ , we will denote  $\text{Op}_0(a)$  by  $a(x, D)$ . When  $a \in \mathcal{S}^*(\mathbb{R}^{2d})$ ,

$$\text{Op}_\tau(a)u(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} e^{i(x-y)\xi} a((1-\tau)x + \tau y, \xi) u(y) dy d\xi, \quad (3)$$

where the integral is absolutely convergent.

**Proposition 2.1.** *The correspondence  $a \mapsto K_\tau$  is an isomorphism of  $\mathcal{S}^*(\mathbb{R}^{2d})$ , of  $\mathcal{S}'^*(\mathbb{R}^{2d})$  and of  $L^2(\mathbb{R}^{2d})$ . The inverse map is given by*

$$a(x, \xi) = \mathcal{F}_{y \rightarrow \xi} K_\tau(x + \tau y, x - (1-\tau)y).$$

Operators with symbols in  $\mathcal{S}^*$  correspond to kernels in  $\mathcal{S}^*$  and by proposition 1.2, those extend to continuous operators from  $\mathcal{S}'^*$  to  $\mathcal{S}^*$ . We will call these \*-regularizing operators.

Let  $A_p$  and  $B_p$  be sequences that satisfy (M.1), (M.3)' and  $A_0 = 1$  and  $B_0 = 1$ . Moreover, let  $A_p \subset M_p$  and  $B_p \subset M_p$  i.e. there exist  $c_0 > 0$  and  $L > 0$  such that  $A_p \leq c_0 L^p M_p$  and  $B_p \leq c_0 L^p M_p$ , for all  $p \in \mathbb{N}$  (it is obvious that without losing generality we can assume that this  $c_0$  is the same with  $c_0$  from the conditions (M.2) and (M.3) for  $M_p$ ). For  $0 < \rho \leq 1$ , define  $\Gamma_{A_p, B_p, \rho}^{M_p, \infty}(\mathbb{R}^{2d}; h, m)$  as the space of all  $a \in \mathcal{C}^\infty(\mathbb{R}^{2d})$  for which the following norm is finite

$$\|a\|_{h, m, \Gamma} = \sup_{\alpha, \beta} \sup_{(x, \xi) \in \mathbb{R}^{2d}} \frac{|D_\xi^\alpha D_x^\beta a(x, \xi)| \langle (x, \xi) \rangle^{\rho|\alpha| + \rho|\beta|} e^{-M(m|\xi|)} e^{-M(m|x|)}}{h^{|\alpha| + |\beta|} A_\alpha B_\beta}.$$

It is easily verified that it is a Banach space. Define

$$\begin{aligned} \Gamma_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}; m) &= \varprojlim_{h \rightarrow 0} \Gamma_{A_p, B_p, \rho}^{M_p, \infty}(\mathbb{R}^{2d}; h, m), \quad \Gamma_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}) = \varinjlim_{m \rightarrow \infty} \Gamma_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}; m), \\ \Gamma_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{2d}; h) &= \varprojlim_{m \rightarrow 0} \Gamma_{A_p, B_p, \rho}^{M_p, \infty}(\mathbb{R}^{2d}; h, m), \quad \Gamma_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{2d}) = \varinjlim_{h \rightarrow \infty} \Gamma_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{2d}; h). \end{aligned}$$

$\Gamma_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}; m)$  and  $\Gamma_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{2d}; h)$  are  $(F)$ -spaces.  $\Gamma_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d})$  and  $\Gamma_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{2d})$  are barreled and bornological locally convex spaces.

**Theorem 2.1.** *Let  $a \in \Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ . Then the integral (3) is well defined as an iterated integral. The ultradistribution  $\text{Op}_\tau(a)u$ ,  $u \in \mathcal{S}^*$ , coincides with the function defined by that iterated integral.*

**Theorem 2.2.** *The mapping  $(a, u) \mapsto \text{Op}_\tau(a)u$ ,  $\Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d}) \times \mathcal{S}^*(\mathbb{R}^d) \longrightarrow \mathcal{S}^*(\mathbb{R}^d)$ , is hypocontinuous.*

Let  $\rho_1 = \inf\{\rho \in \mathbb{R}_+ | A_p \subset M_p^\rho\}$  and  $\rho_2 = \inf\{\rho \in \mathbb{R}_+ | B_p \subset M_p^\rho\}$  and put  $\rho_0 = \max\{\rho_1, \rho_2\}$ . Then  $0 < \rho_0 \leq 1$  and for every  $\rho$  such that  $\rho_0 \leq \rho \leq 1$ , if the larger infimum can be reached, or, otherwise  $\rho_0 < \rho \leq 1$ ,  $A_p \subset M_p^\rho$  and  $B_p \subset M_p^\rho$ . So, for every such  $\rho$ , there exists  $c'_0 > 0$  and  $L > 0$  (which depend on  $\rho$ ) such that,  $A_p \leq c'_0 L^p M_p^\rho$ ,  $B_p \leq c'_0 L^p M_p^\rho$ . Moreover, because  $M_p$  tends to infinity, there exists  $\tilde{c} > 0$  such that  $M_p^\rho \leq \tilde{c} M_p$ , for all such  $\rho$ . From now on we suppose that  $\rho_0 \leq \rho \leq 1$ , if the larger infimum can be reached, or otherwise  $\rho_0 < \rho \leq 1$ .

For  $t > 0$ , put  $Q_t = \{(x, \xi) \in \mathbb{R}^{2d} | \langle x \rangle < t, \langle \xi \rangle < t\}$  and  $Q_t^c = \mathbb{R}^{2d} \setminus Q_t$ . Denote by  $FS_{A_p, B_p, \rho}^{M_p, \infty}(\mathbb{R}^{2d}; B, h, m)$  the vector space of all formal series  $\sum_{j=0}^{\infty} a_j(x, \xi)$  such that  $a_j \in \mathcal{C}^\infty(\text{int } Q_{Bm_j}^c)$ ,  $D_\xi^\alpha D_x^\beta a_j(x, \xi)$  can be extended to continuous function on  $Q_{Bm_j}^c$  for all  $\alpha, \beta \in \mathbb{N}^d$  and

$$\sup_{j \in \mathbb{N}} \sup_{\alpha, \beta} \sup_{(x, \xi) \in Q_{Bm_j}^c} \frac{|D_\xi^\alpha D_x^\beta a_j(x, \xi)| \langle (x, \xi) \rangle^{\rho|\alpha| + \rho|\beta| + 2j\rho} e^{-M(m|\xi|)} e^{-M(m|x|)}}{h^{|\alpha| + |\beta| + 2j} A_\alpha B_\beta A_j B_j} < \infty.$$

In the above, we use the convention  $m_0 = 0$  and hence  $Q_{B_{m_0}}^c = \mathbb{R}^{2d}$ . It is easy to check that  $FS_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}; B, h, m)$  is a Banach space. Define

$$\begin{aligned} FS_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}; B, m) &= \varprojlim_{h \rightarrow 0} FS_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}; B, h, m), \\ FS_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}) &= \varinjlim_{B, m \rightarrow \infty} FS_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}; B, m), \\ FS_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{2d}; B, h) &= \varprojlim_{m \rightarrow 0} FS_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}; B, h), \\ FS_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{2d}) &= \varinjlim_{B, h \rightarrow \infty} FS_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{2d}; B, h). \end{aligned}$$

$FS_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}; B, m)$  and  $FS_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{2d}; B, h)$  are  $(F)$  - spaces.  $FS_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d})$  and  $FS_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{2d})$  are barreled and bornological locally convex spaces. Note, also, that the inclusions  $\Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d}) \longrightarrow FS_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ , defined as  $a \mapsto \sum_{j \in \mathbb{N}} a_j$ , where  $a_0 = a$  and  $a_j = 0$ ,  $j \geq 1$ , is continuous.

**Definition 2.1.** Two sums,  $\sum_{j \in \mathbb{N}} a_j, \sum_{j \in \mathbb{N}} b_j \in FS_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ , are said to be equivalent, in notation  $\sum_{j \in \mathbb{N}} a_j \sim \sum_{j \in \mathbb{N}} b_j$ , if there exist  $m > 0$  and  $B > 0$ , resp. there exist  $h > 0$  and  $B > 0$ , such that for every  $h > 0$ , resp. for every  $m > 0$ ,

$$\sup_{N \in \mathbb{Z}_+} \sup_{\alpha, \beta} \sup_{(x, \xi) \in Q_{B_{m_N}}^c} \frac{\left| D_\xi^\alpha D_x^\beta \sum_{j < N} (a_j(x, \xi) - b_j(x, \xi)) \right| \langle (x, \xi) \rangle^{\rho|\alpha| + \rho|\beta| + 2N\rho}}{h^{|\alpha| + |\beta| + 2N} A_\alpha B_\beta A_N B_N} \cdot e^{-M(m|\xi|)} e^{-M(m|x|)} < \infty.$$

From now on, we assume that  $A_p$  and  $B_p$  satisfy (M.2). Without losing generality we can assume that the constants  $c_0$  and  $H$  from the condition (M.2) for  $A_p$  and  $B_p$  are the same as the corresponding constants for  $M_p$ .

**Theorem 2.3.** Let  $a \in \Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$  be such that  $a \sim 0$ . Then, for every  $\tau \in \mathbb{R}$ ,  $\text{Op}_\tau(a)$  is  $*$ -regularizing.

**Theorem 2.4.** Let  $\sum_{j \in \mathbb{N}} a_j \in FS_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$  be given. Then, there exists  $a \in \Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ , such that  $a \sim \sum_{j \in \mathbb{N}} a_j$ .

In the proof of the theorem the construction of  $a$  is given in the following way. Let  $\varphi(x) \in \mathcal{D}^{(B_p)}(\mathbb{R}^d)$  and  $\psi(\xi) \in \mathcal{D}^{(A_p)}(\mathbb{R}^d)$ , in the  $(M_p)$  case, resp.  $\varphi(x) \in \mathcal{D}^{\{B_p\}}(\mathbb{R}^d)$  and  $\psi(\xi) \in \mathcal{D}^{\{A_p\}}(\mathbb{R}^d)$  in the  $\{M_p\}$  case, are such that  $0 \leq \varphi, \psi \leq 1$ ,  $\varphi(x) = 1$  when



$\langle x \rangle \leq 2$ ,  $\psi(\xi) = 1$  when  $\langle \xi \rangle \leq 2$  and  $\varphi(x) = 0$  when  $\langle x \rangle \geq 3$ ,  $\psi(\xi) = 0$  when  $\langle \xi \rangle \geq 3$ . Put  $\chi(x, \xi) = \varphi(x)\psi(\xi)$ ,  $\chi_n(x, \xi) = \chi\left(\frac{x}{Rm_n}, \frac{\xi}{Rm_n}\right)$  for  $n \in \mathbb{Z}_+$  and  $R > 0$  and put  $\chi_0(x, \xi) = 0$ . The desired  $a$  can be define to be  $a(x, \xi) = \sum_j (1 - \chi_j(x, \xi)) a_j(x, \xi)$  for sufficiently large  $R$  in the definition of  $\chi_n$ .

**Theorem 2.5.** *Let  $\tau, \tau_1 \in \mathbb{R}$  and  $a \in \Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ . There exists  $b \in \Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$  and  $*$ -regularizing operator  $T$  such that  $\text{Op}_{\tau_1}(a) = \text{Op}_{\tau}(b) + T$ . Moreover,*

$$b(x, \xi) \sim \sum_{\beta} \frac{1}{\beta!} (\tau_1 - \tau)^{|\beta|} \partial_{\xi}^{\beta} D_x^{\beta} a(x, \xi), \text{ in } FS_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d}).$$

**Theorem 2.6.** *Let  $\tau \in \mathbb{R}$  and  $a \in \Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ . The transposed operator,  ${}^t\text{Op}_{\tau}(a)$ , is still a pseudo-differential operator and it is equal to  $\text{Op}_{1-\tau}(a(x, -\xi))$ . Moreover, there exist  $b \in \Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$  and  $*$ -regularizing operator  $T$  such that  ${}^t\text{Op}_{\tau}(a) = \text{Op}_{\tau}(b) + T$  and*

$$b(x, \xi) \sim \sum_{\alpha} \frac{1}{\alpha!} (1 - 2\tau)^{|\alpha|} (-\partial_{\xi})^{\alpha} D_x^{\alpha} a(x, -\xi) \text{ in } FS_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d}).$$

**Theorem 2.7.** *Let  $a, b \in \Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ . There exist  $f \in \Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$  and  $*$ -regularizing operator  $T$  such that  $a(x, D)b(x, D) = f(x, D) + T$  and  $f$  has the asymptotic expansion*

$$f(x, \xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} a(x, \xi) D_x^{\alpha} b(x, \xi) \text{ in } FS_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d}).$$

We end this section with the following technical lemma which is also proven in [16].

**Lemma 2.1.** *Let  $M_p$  be a sequence which satisfies (M.1), (M.2) and (M.3) and  $m$  a positive real. Then, for all  $n \in \mathbb{Z}_+$ ,  $M(mm_n) \leq 2(c_0 m + 2)n \ln H + \ln c_0$ , where  $c_0$  is the constant from the conditions (M.2) and (M.3). If  $(t_p) \in \mathfrak{R}$  then,  $N_{t_p}(mm_n) \leq n \ln H + \ln c$  for all  $n \in \mathbb{Z}_+$ , where the constant  $c$  depends only on  $M_p$ ,  $(t_p)$  and  $m$ , but not on  $n$ .*

### 3 Anti-Wick quantization

First we recall definition and some basic facts about the short-time Fourier transform. It will be convenient to introduce some notation to make the definitions less cumbersome. Put  $\mathcal{G}_0(x) = \pi^{-d/4} e^{-\frac{1}{2}|x|^2}$  and  $\mathcal{G}_{y, \eta}(x) = \pi^{-d/4} e^{ix\eta} e^{-\frac{1}{2}|x-y|^2}$ , where  $y$  and  $\eta$  are parameters in  $\mathbb{R}^d$  and denote by  $(\cdot, \cdot)$  the inner product in  $L^2$ .

**Definition 3.1.** *For  $u \in \mathcal{S}'^*$  we define the short-time Fourier transform  $Vu$  of  $u$  as the tempered ultradistribution in  $\mathbb{R}^{2d}$  given by  $Vu(y, \eta) = \mathcal{F}_{t \rightarrow \eta}(u(t)\mathcal{G}_0(t - y))$ .*

**Proposition 3.1.** *The short-time Fourier transform acts continuously  $\mathcal{S}'^*(\mathbb{R}^d) \longrightarrow \mathcal{S}'^*(\mathbb{R}^{2d})$ ,  $\mathcal{S}^*(\mathbb{R}^d) \longrightarrow \mathcal{S}^*(\mathbb{R}^{2d})$  and  $L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^{2d})$ . Moreover  $\|Vu\|_{L^2(\mathbb{R}^{2d})} = (2\pi)^{d/2} \|u\|_{L^2(\mathbb{R}^d)}$ .*



Its adjoint map  $V^* : \mathcal{S}^*(\mathbb{R}^{2d}) \longrightarrow \mathcal{S}^*(\mathbb{R}^d)$ ,

$$V^*F(t) = (2\pi)^d \int_{\mathbb{R}^d} \mathcal{F}_{\eta \rightarrow t}^{-1}(F(y, \eta)) \mathcal{G}_0(t - y) dy, \quad F \in \mathcal{S}^*(\mathbb{R}^{2d})$$

extends to a well defined and continuous map  $\mathcal{S}'^*(\mathbb{R}^{2d}) \longrightarrow \mathcal{S}'^*(\mathbb{R}^d)$  and  $L^2(\mathbb{R}^{2d}) \longrightarrow L^2(\mathbb{R}^d)$  and  $V^*V = (2\pi)^d I$ . Now we can define Anti-Wick operators.

**Definition 3.2.** Let  $a \in \mathcal{S}'^*(\mathbb{R}^{2d})$ . We define the Anti-Wick operator with symbol  $a$  as the map  $A_a : \mathcal{S}^*(\mathbb{R}^d) \longrightarrow \mathcal{S}'^*(\mathbb{R}^d)$  given by  $A_a u = (2\pi)^{-d} V^*(aVu)$ ,  $u \in \mathcal{S}^*(\mathbb{R}^d)$ .

Observe that, if  $a$  is a multiplier for  $\mathcal{S}^*(\mathbb{R}^{2d})$  (for example an element of  $\Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ ), then  $A_a$  maps  $\mathcal{S}^*(\mathbb{R}^d)$  continuously into itself. Also, note that the above formula is equivalent to

$$\langle A_a u, \bar{v} \rangle = (2\pi)^{-d} \langle a, Vu\bar{V}v \rangle, \quad u, v \in \mathcal{S}^*(\mathbb{R}^d). \quad (4)$$

From this, the following propositions follow.

**Proposition 3.2.** Let  $a_n \in \mathcal{S}'^*(\mathbb{R}^{2d})$  be a sequence that converges to  $a$  in  $\mathcal{S}'^*(\mathbb{R}^{2d})$ , then  $A_{a_n} u \longrightarrow A_a u$ , for every  $u \in \mathcal{S}^*(\mathbb{R}^d)$ .

**Proposition 3.3.** Let  $a \in \mathcal{S}'^*(\mathbb{R}^{2d})$  be real valued. Then  $A_a$  is formally self-adjoint.

If  $a$  is locally integrable function of \*-ultrapolynomial growth (for example, if it is an element of  $\Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ ), then, by (4), we can represent the action of  $A_a$  as

$$A_a u(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} a(y, \eta) (u, \mathcal{G}_{y, \eta}) \mathcal{G}_{y, \eta}(x) dy d\eta, \quad u \in \mathcal{S}^*(\mathbb{R}^d).$$

The proof of the following proposition is the same as in the case of distribution and it will be omitted (see for example [10]).

**Proposition 3.4.** Let  $a \in \mathcal{S}'^*(\mathbb{R}^{2d})$ . Then  $A_a = b^w$  where  $b \in \mathcal{S}'^*(\mathbb{R}^{2d})$  is given by

$$b(x, \xi) = \pi^{-d} \left( a(\cdot, \cdot) * e^{-|\cdot|^2 - |\cdot|^2} \right) (x, \xi). \quad (5)$$

From now on we assume that  $A_p = B_p$ . Our goal is to represent the Anti-Wick operator  $A_a$ , for  $a \in \Gamma_{A_p, A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$  as a pseudo-differential operator  $b^w$  for some  $b \in \Gamma_{A_p, A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ . First, note that  $|\eta|^{2k} \leq k! e^{|\eta|^2}$ , for all  $k \in \mathbb{N}$ . From this one easily obtains the following inequality

$$\langle \eta \rangle^k \leq 2^k \sqrt{k!} e^{|\eta|^2/2}. \quad (6)$$

**Theorem 3.1.** *Let  $a \in \Gamma_{A_p, A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ . Then there exists  $\tilde{b} \in \Gamma_{A_p, A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$  and  $*$ -regularizing operator  $T$  such that  $A_a = \tilde{b}^w + T$ . Moreover,  $\tilde{b}$  has an asymptotic expansion  $\sum_j p_j$  in  $FS_{A_p, A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ , where  $p_0 = a(x, \xi)$  and*

$$p_j(x, \xi) = \sum_{2j-1 \leq |\alpha+\beta| \leq 2j} \frac{c_{\alpha, \beta}}{\alpha! \beta!} \partial_\xi^\alpha \partial_x^\beta a(x, \xi)$$

$$\text{where } c_{\alpha, \beta} = \frac{1}{\pi^d} \int_{\mathbb{R}^{2d}} \eta^\alpha y^\beta e^{-|y|^2 - |\eta|^2} dy d\eta.$$

*Proof.* First we will prove that  $\sum_j p_j \in FS_{A_p, A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ . Note that  $c_{\alpha, \beta} = 0$  if  $|\alpha + \beta|$  is odd. Hence

$$p_j(x, \xi) = \sum_{|\alpha+\beta|=2j} \frac{c_{\alpha, \beta}}{\alpha! \beta!} \partial_\xi^\alpha \partial_x^\beta a(x, \xi).$$

If we use the fact  $|\eta|^k \leq \sqrt{k!} e^{|\eta|^2/2}$  we have  $|c_{\alpha, \beta}| \leq c' \sqrt{|\alpha!| |\beta!|}$ , where we put  $c' = \frac{1}{\pi^d} \int_{\mathbb{R}^{2d}} e^{-|y|^2/2 - |\eta|^2/2} dy d\eta$ . For the derivatives of  $p_j$  we have

$$\begin{aligned} |D_\xi^\gamma D_x^\delta p_j(x, \xi)| &\leq C'_1 \sum_{|\alpha+\beta|=2j} \frac{|c_{\alpha, \beta}|}{\alpha! \beta!} \cdot \frac{h^{|\gamma|+|\delta|+2j} A_{\alpha+\gamma} A_{\beta+\delta} e^{M(m|\xi|)} e^{M(m|x|)}}{\langle (x, \xi) \rangle^{\rho|\gamma|+\rho|\delta|+2\rho j}} \\ &\leq C_1 \sum_{|\alpha+\beta|=2j} \frac{d^{2j}}{\sqrt{|\alpha!| |\beta!|}} \cdot \frac{(hH)^{|\gamma|+|\delta|+2j} A_\gamma A_\delta A_{2j} e^{M(m|\xi|)} e^{M(m|x|)}}{\langle (x, \xi) \rangle^{\rho|\gamma|+\rho|\delta|+2\rho j}} \\ &\leq C_2 2^{2j+2d-1} \frac{(hH)^{|\gamma|+|\delta|+2j} (dH)^{2j} A_\gamma A_\delta A_j A_j e^{M(m|\xi|)} e^{M(m|x|)}}{\langle (x, \xi) \rangle^{\rho|\gamma|+\rho|\delta|+2\rho j}}, \end{aligned}$$

i.e., we obtain  $\frac{|D_\xi^\gamma D_x^\delta p_j(x, \xi)| \langle (x, \xi) \rangle^{\rho|\gamma|+\rho|\delta|+2\rho j} e^{-M(m|\xi|)} e^{-M(m|x|)}}{(2dhH^2)^{|\gamma|+|\delta|+2j} A_\gamma A_\delta A_j A_j} \leq C$ , for all  $(x, \xi) \in \mathbb{R}^{2d}$ ,  $\gamma, \delta \in \mathbb{N}$ ,  $j \in \mathbb{N}$ . Hence  $\sum_j p_j \in FS_{A_p, A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ . Take  $\chi_j$  as in the remark after theorem 2.4 and define  $\tilde{b} = \sum_j (1 - \chi_j) p_j$ . Then  $\tilde{b} \in \Gamma_{A_p, A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$  and  $\tilde{b} \sim \sum_j p_j$ . It is enough to prove that  $b - \tilde{b} \in \mathcal{S}^*$ , for  $b$  defined as in (5). We have

$$b(x, \xi) - \tilde{b}(x, \xi) = \chi_0(x, \xi) b(x, \xi) + \sum_{n=0}^{\infty} (\chi_{n+1} - \chi_n)(x, \xi) \left( b(x, \xi) - \sum_{j=0}^n p_j(x, \xi) \right).$$

By definition,  $\chi_0 = 0$ . We Taylor expand  $a$  and we obtain

$$a(y, \eta) = \sum_{|\alpha|+|\beta| \leq 2n+1} \frac{1}{\alpha! \beta!} \partial_\xi^\alpha \partial_x^\beta a(x, \xi) (\eta - \xi)^\alpha (y - x)^\beta + r_{2n+2}(x, y, \xi, \eta),$$

where  $r_{2n+2}$  is the reminder

$$r_{2n+2}(x, y, \xi, \eta) = (2n+2) \sum_{|\alpha+\beta|=2n+2} \frac{1}{\alpha! \beta!} (\eta - \xi)^\alpha (y - x)^\beta \cdot \int_0^1 (1-t)^{2n+1} \partial_\xi^\alpha \partial_x^\beta a(x + t(y-x), \xi + t(\eta - \xi)) dt.$$

If we put this in the expression for  $b - \tilde{b}$ , keeping in mind the way we defined  $p_j$ , we obtain

$$b(x, \xi) - \tilde{b}(x, \xi) = \frac{1}{\pi^d} \sum_{n=0}^{\infty} (\chi_{n+1} - \chi_n)(x, \xi) \sum_{|\alpha+\beta|=2n+2} \frac{2n+2}{\alpha! \beta!} I_{\alpha, \beta}(x, \xi),$$

where we put

$$I_{\alpha, \beta}(x, \xi) = \int_0^1 \int_{\mathbb{R}^{2d}} \eta^\alpha y^\beta (1-t)^{2n+1} \partial_\xi^\alpha \partial_x^\beta a(x + ty, \xi + t\eta) e^{-|y|^2 - |\eta|^2} dy d\eta dt.$$

We will estimate the derivatives of  $I_{\alpha, \beta}$ .

$$\begin{aligned} & |\partial_\xi^\gamma \partial_x^\delta I_{\alpha, \beta}(x, \xi)| \\ & \leq \int_0^1 \int_{\mathbb{R}^{2d}} |\eta|^{|\alpha|} |y|^{|\beta|} |\partial_\xi^{\alpha+\gamma} \partial_x^{\beta+\delta} a(x + ty, \xi + t\eta)| e^{-|y|^2 - |\eta|^2} dy d\eta dt \\ & \leq C'_1 \int_0^1 \int_{\mathbb{R}^{2d}} |\eta|^{|\alpha|} |y|^{|\beta|} \frac{h^{|\gamma|+|\delta|+2n+2} A_{\alpha+\gamma} A_{\beta+\delta} e^{M(m|\xi+t\eta|)} e^{M(m|x+ty|)}}{\langle (x + ty, \xi + t\eta) \rangle^{\rho|\gamma|+\rho|\delta|+(2n+2)\rho}} e^{-|y|^2 - |\eta|^2} dy d\eta dt \\ & \leq C'_1 \int_0^1 \int_{\mathbb{R}^{2d}} \frac{h^{|\gamma|+|\delta|+2n+2} A_{\gamma+\delta+2n+2} \langle (y, \eta) \rangle^{2n+2} e^{M(m|\xi+t\eta|)} e^{M(m|x+ty|)}}{\langle (x + ty, \xi + t\eta) \rangle^{(2n+2)\rho} e^{|y|^2 + |\eta|^2}} dy d\eta dt \\ & \leq C''_1 \int_0^1 \int_{\mathbb{R}^{2d}} \frac{(2hL)^{|\gamma|+|\delta|+2n+2} M_{\gamma+\delta+2n+2}^\rho \langle (y, \eta) \rangle^{4n+4} e^{M(m|\xi+t\eta|)} e^{M(m|x+ty|)}}{\langle (x, \xi) \rangle^{(2n+2)\rho} e^{|y|^2 + |\eta|^2}} dy d\eta dt \\ & \leq C_1 \frac{\sqrt{(4n+4)!} (8hLH)^{|\gamma|+|\delta|+2n+2} M_{\gamma+\delta}^\rho M_{2n+2}^\rho}{\langle (x, \xi) \rangle^{(2n+2)\rho}} \int_0^1 \int_{\mathbb{R}^{2d}} \frac{e^{M(m|\xi+t\eta|)} e^{M(m|x+ty|)}}{e^{|y|^2/2 + |\eta|^2/2}} dy d\eta dt, \end{aligned}$$

where, in the last inequality, we used (6). For shorter notations, we will denote the last integral by  $\tilde{I}(x, \xi)$ . Note that  $\langle (x, \xi) \rangle \geq Rm_n$  on the support of  $\chi_{n+1} - \chi_n$ . For the derivatives of  $(\chi_{n+1} - \chi_n)(x, \xi) I_{\alpha, \beta}(x, \xi)$ , we have

$$\begin{aligned} & |\partial_\xi^\gamma \partial_x^\delta ((\chi_{n+1} - \chi_n)(x, \xi) I_{\alpha, \beta}(x, \xi))| \\ & \leq \sum_{\substack{\gamma' \leq \gamma \\ \delta' \leq \delta}} \binom{\gamma}{\gamma'} \binom{\delta}{\delta'} |\partial_\xi^{\gamma-\gamma'} \partial_x^{\delta-\delta'} ((\chi_{n+1} - \chi_n)(x, \xi))| |\partial_\xi^{\gamma'} \partial_x^{\delta'} I_{\alpha, \beta}(x, \xi)| \\ & \leq C_2 \sum_{\substack{\gamma' \leq \gamma \\ \delta' \leq \delta}} \binom{\gamma}{\gamma'} \binom{\delta}{\delta'} \frac{h_1^{|\gamma|-|\gamma'|+|\delta|-|\delta'|} A_{\gamma-\gamma'} A_{\delta-\delta'}}{(Rm_n)^{|\gamma|-|\gamma'|+|\delta|-|\delta'|}} \end{aligned}$$

$$\begin{aligned}
& \cdot \frac{\sqrt{(4n+4)!}(8hLH)^{|\gamma'|+|\delta'|+2n+2} M_{\gamma'+\delta'} M_{2n+2}^\rho}{(Rm_n)^{(2n+2)\rho}} \cdot \tilde{I}(x, \xi) \\
& \leq C_3 \sum_{\substack{\gamma' \leq \gamma \\ \delta' \leq \delta}} \binom{\gamma}{\gamma'} \binom{\delta}{\delta'} \frac{(h_1 L)^{|\gamma|-|\gamma'|+|\delta|-|\delta'|}}{(RM_1)^{|\gamma|-|\gamma'|+|\delta|-|\delta'|}} \\
& \quad \cdot \frac{\sqrt{(4n+4)!}(8hLH)^{|\gamma'|+|\delta'|+2n+2} H^{2n+2} M_{\gamma+\delta} M_{n+1}^{2\rho}}{(Rm_n)^{(2n+2)\rho}} \cdot \tilde{I}(x, \xi) \\
& \leq C_4 \left( \frac{h_1 L}{RM_1} + 8hLH \right)^{|\gamma|+|\delta|} \frac{\sqrt{(4n+4)!}(8hLH^3)^{2n+2} M_{\gamma+\delta} M_n^{2\rho}}{R^{(2n+2)\rho} m_n^{(2n+2)\rho}} \cdot \tilde{I}(x, \xi) \\
& \leq C'_4 \left( \frac{h_1 L}{RM_1} + 8hLH \right)^{|\gamma|+|\delta|} \frac{\sqrt{(4n+4)!}(8hLH^3)^{2n+2} M_{\gamma+\delta}}{R^{(2n+2)\rho}} \cdot \tilde{I}(x, \xi),
\end{aligned}$$

where, in the last inequality, we used that

$$m_n^{n+1} \geq m_n \cdot \dots \cdot m_2 \cdot m_1 \cdot m_1 = M_n M_1.$$

Let  $m' > 0$  be arbitrary but fixed. Then one easily proves that  $e^{M(m'|(x,\xi)|)} \leq e^{M(m'(|x|+|\xi|))} \leq 2e^{M(2m'|x|)} e^{M(2m'|\xi|)}$  (one easily proves that  $e^{M(\lambda+\nu)} \leq 2e^{M(2\lambda)} e^{M(2\nu)}$ ). Then we have

$$\begin{aligned}
e^{M(m|\xi+t\eta|)} &= e^{-M(2m'|\xi|)} e^{M(2m'|\xi|)} e^{M(m|\xi+t\eta|)} \leq 2e^{-M(2m'|\xi|)} e^{M(4m'|t\eta|)} e^{M(4m'|\xi+t\eta|)} e^{M(m|\xi+t\eta|)} \\
&\leq c_1 e^{-M(2m'|\xi|)} e^{M(4m'|\eta|)} e^{M((m+4m')H|\xi+t\eta|)},
\end{aligned}$$

where, in the last inequality, we used proposition 3.6 of [6]. Similarly

$$e^{M(m|x+ty|)} \leq c_1 e^{-M(2m'|x|)} e^{M(4m'|y|)} e^{M((m+4m')H|x+ty|)}.$$

Obviously  $e^{M(4m'|\eta|)} \leq c_2 e^{|\eta|^2/4}$  and  $e^{M(4m'|y|)} \leq c_2 e^{|y|^2/4}$  for some  $c_2 > 0$  which depends only on  $M_p$  and  $m'$ . We obtain

$$\tilde{I}(x, \xi) \leq c_3 e^{-M(m'|(x,\xi)|)} \int_0^1 \left( \int_{\mathbb{R}^d} \frac{e^{M((m+4m')H|x+ty|)}}{e^{|y|^2/4}} dy \cdot \int_{\mathbb{R}^d} \frac{e^{M((m+4m')H|\xi+t\eta|)}}{e^{|\eta|^2/4}} d\eta \right) dt.$$

Note that, when  $|y| \leq |x|$  we have  $e^{M((m+4m')H|x+ty|)} \leq e^{M(2(m+4m')H|x|)} \leq e^{M(6(m+4m')HRm_{n+1})}$ , on the support of  $\chi_{n+1} - \chi_n$  (where  $|x| \leq 3Rm_{n+1}$ ). When  $|y| > |x|$  we have  $e^{M((m+4m')H|x+ty|)} \leq e^{M(2(m+4m')H|y|)} \leq c_4 e^{|y|^2/8}$ , for some  $c_4 > 0$ . We obtain

$$\begin{aligned}
& \int_{\mathbb{R}^d} \frac{e^{M((m+4m')H|x+ty|)}}{e^{|y|^2/4}} dy \\
&= \int_{|y| \leq |x|} \frac{e^{M((m+4m')H|x+ty|)}}{e^{|y|^2/4}} dy + \int_{|y| > |x|} \frac{e^{M((m+4m')H|x+ty|)}}{e^{|y|^2/4}} dy \\
&\leq e^{M(6(m+4m')HRm_{n+1})} \int_{|y| \leq |x|} \frac{1}{e^{|y|^2/4}} dy + c_4 \int_{|y| > |x|} \frac{1}{e^{|y|^2/8}} dy \leq c_5 e^{M(6(m+4m')HRm_{n+1})}.
\end{aligned}$$

We can obtain similar estimate for the other integral. By lemma 2.1, we have

$$e^{M(6(m+4m')HRm_{n+1})} \leq c_0 H^{2(n+1)(6c_0(m+4m')HR+2)}.$$

So, we have

$$\tilde{I}(x, \xi) \leq c_6 e^{-M(m'|(x, \xi)|)} e^{2M(6(m+4m')HRm_{n+1})} \leq c_7 e^{-M(m'|(x, \xi)|)} H^{4(n+1)(6c_0(m+4m')HR+2)}$$

on the support of  $\chi_{n+1} - \chi_n$ . If we insert this in the estimates for the derivatives of the terms  $(\chi_{n+1} - \chi_n)(x, \xi) I_{\alpha, \beta}(x, \xi)$ , we obtain

$$\begin{aligned} & \left| \partial_\xi^\gamma \partial_x^\delta ((\chi_{n+1} - \chi_n)(x, \xi) I_{\alpha, \beta}(x, \xi)) \right| \\ & \leq C_5 \left( \frac{h_1 L}{RM_1} + 8hLH \right)^{|\gamma|+|\delta|} \frac{\sqrt{(4n+4)!} (8hLH^3)^{2n+2} M_{\gamma+\delta}}{R^{(2n+2)\rho}} \\ & \quad \cdot e^{-M(m'|(x, \xi)|)} H^{4(n+1)(6c_0(m+4m')HR+2)}. \end{aligned}$$

We will consider first the  $(M_p)$  case. Take  $R$  such that  $RM_1 \geq L$  and  $32d/R^\rho \leq 1/2$ . Then choose  $h_1$  such that  $h_1 \leq 1/(2m')$  and  $h$  such that  $8hLH^{3+2(6c_0(m+4m')HR+2)} \leq 1$  and  $8hLH \leq 1/(2m')$ . Note that, the choice of  $R$  (and hence  $\chi_j$ ) doesn't depend on  $m'$ , only on  $A_p$ ,  $M_p$  and  $a$ . For  $|\alpha + \beta| = 2n + 2$ , we have

$$\alpha! \beta! \geq \frac{|\alpha|! |\beta|!}{d^{2n+2}} \geq \frac{(2n+2)!}{(2d)^{2n+2}}.$$

Also,  $\sqrt{(4n+4)!} \leq 2^{2n+2} (2n+2)!$ . Now we obtain

$$\begin{aligned} & \sum_{|\alpha+\beta|=2n+2} \frac{2n+2}{\alpha! \beta!} \left| \partial_\xi^\gamma \partial_x^\delta ((\chi_{n+1} - \chi_n)(x, \xi) I_{\alpha, \beta}(x, \xi)) \right| \\ & \leq C_5 \sum_{|\alpha+\beta|=2n+2} \frac{2^{2n+2} (2d)^{2n+2}}{(2n+2)!} \left( \frac{h_1 L}{RM_1} + 8hLH \right)^{|\gamma|+|\delta|} \frac{2^{2n+2} (2n+2)! M_{\gamma+\delta}}{R^{(2n+2)\rho}} e^{-M(m'|(x, \xi)|)} \\ & \leq C_5 e^{-M(m'|(x, \xi)|)} \frac{M_{\gamma+\delta}}{m'^{|\gamma|+|\delta|}} \cdot \left( \frac{8d}{R^\rho} \right)^{2n+2} \cdot 2^{2n+2+2d-1} \leq C_6 \frac{M_{\gamma+\delta}}{m'^{|\gamma|+|\delta|}} e^{-M(m'|(x, \xi)|)} \cdot \frac{1}{4^{2n+2}}, \end{aligned}$$

where, in the last inequality, we put  $C_6 = 2^{2d-1} C_5$ . Hence, for the derivatives of

$$\sum_{n=0}^{\infty} (\chi_{n+1} - \chi_n)(x, \xi) \sum_{|\alpha+\beta|=2n+2} \frac{2n+2}{\alpha! \beta!} I_{\alpha, \beta}(x, \xi),$$

we obtain the estimate  $C \frac{M_{\gamma+\delta}}{m'^{|\gamma|+|\delta|}} e^{-M(m'|(x, \xi)|)}$  and by the arbitrariness of  $m'$ , it follows that

it is a  $\mathcal{S}^{(M_p)}$  function. Let us consider the  $\{M_p\}$  case. Take  $R$  such that  $\frac{256dhLH^9}{R^\rho} \leq \frac{1}{2}$ .

Then, choose  $m$  and  $m'$  such that  $6c_0(m + 4m')HR \leq 1$ . Then we have

$$\begin{aligned}
& \left| \partial_\xi^\gamma \partial_x^\delta ((\chi_{n+1} - \chi_n)(x, \xi) I_{\alpha, \beta}(x, \xi)) \right| \\
& \leq C_5 \left( \frac{h_1 L}{RM_1} + 8hLH \right)^{|\gamma|+|\delta|} \frac{\sqrt{(4n+4)!} (8hLH^3)^{2n+2} M_{\gamma+\delta}}{R^{(2n+2)\rho}} \\
& \quad \cdot e^{-M(m'|(x, \xi)|)} H^{4(n+1)(6c_0(m+4m')HR+2)} \\
& \leq C_5 \left( \frac{h_1 L}{RM_1} + 8hLH \right)^{|\gamma|+|\delta|} \frac{\sqrt{(4n+4)!} (8hLH^3)^{2n+2} M_{\gamma+\delta}}{R^{(2n+2)\rho}} \cdot e^{-M(m'|(x, \xi)|)} H^{12(n+1)}.
\end{aligned}$$

So

$$\begin{aligned}
& \sum_{|\alpha+\beta|=2n+2} \frac{2n+2}{\alpha! \beta!} \left| \partial_\xi^\gamma \partial_x^\delta ((\chi_{n+1} - \chi_n)(x, \xi) I_{\alpha, \beta}(x, \xi)) \right| \\
& \leq C_5 \sum_{|\alpha+\beta|=2n+2} \left( \frac{h_1 L}{RM_1} + 8hLH \right)^{|\gamma|+|\delta|} \frac{(8d)^{2n+2} (8hLH^9)^{2n+2} M_{\gamma+\delta}}{R^{(2n+2)\rho}} e^{-M(m'|(x, \xi)|)} \\
& \leq C_5 e^{-M(m'|(x, \xi)|)} M_{\gamma+\delta} \left( \frac{h_1 L}{RM_1} + 8hLH \right)^{|\gamma|+|\delta|} \cdot \left( \frac{64dhLH^9}{R^\rho} \right)^{2n+2} \cdot 2^{2n+2+2d-1} \\
& \leq C_6 M_{\gamma+\delta} \left( \frac{h_1 L}{RM_1} + 8hLH \right)^{|\gamma|+|\delta|} e^{-M(m'|(x, \xi)|)} \cdot \frac{1}{4^{2n+2}}.
\end{aligned}$$

Hence, for the derivatives of

$$\sum_{n=0}^{\infty} (\chi_{n+1} - \chi_n)(x, \xi) \sum_{|\alpha+\beta|=2n+2} \frac{2n+2}{\alpha! \beta!} I_{\alpha, \beta}(x, \xi),$$

we obtain the estimate  $CM_{\gamma+\delta} \frac{1}{m''^{|\gamma|+|\delta|}} e^{-M(m'|(x, \xi)|)}$ , where we put  $\frac{1}{m''} = \frac{h_1 L}{RM_1} + 8hLH$ , i.e. it is a  $\mathcal{S}^{\{M_p\}}$  function. In both cases we obtain that  $b - \tilde{b} \in \mathcal{S}^*$ , which completes the proof.  $\square$

Now we want to represent the Weyl quantization of  $b \in \Gamma_{A_p, A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$  by an Anti-Wick operator  $A_a$ , for some  $a \in \Gamma_{A_p, A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ . First we will prove the following technical lemma.

**Lemma 3.1.** *Let  $\sum_k q_k^{(j)} \in FS_{A_p, A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$  for all  $j \in \mathbb{N}$ , such that  $q_0^{(j)} = \dots = q_{j-1}^{(j)} = 0$ . Assume that there exist  $m > 0$  and  $B > 0$ , resp.  $h > 0$  and  $B > 0$ , such that  $\sum_k q_k^{(j)} \in FS_{A_p, A_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}; B, m)$  for all  $j \in \mathbb{N}$ , resp.  $\sum_k q_k^{(j)} \in FS_{A_p, A_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{2d}; B, h)$  for all  $j \in \mathbb{N}$ . Moreover, assume that the constants  $C_{j, h}$ , resp.  $C_{j, m}$ , in*

$$\sup_{k \in \mathbb{N}} \sup_{\alpha, \beta} \sup_{(x, \xi) \in Q_{Bm_k}^c} \frac{\left| D_\xi^\alpha D_x^\beta q_k^{(j)}(x, \xi) \right| \langle (x, \xi) \rangle^{\rho|\alpha| + \rho|\beta| + 2k\rho} e^{-M(m|\xi|)} e^{-M(m|x|)}}{h^{|\alpha|+|\beta|+2k} A_\alpha A_\beta A_k A_k} = C_{j, h} < \infty$$

resp. the same with  $C_{j,m}$  in place of  $C_{j,h}$  in the  $\{M_p\}$  case, are bounded for all  $j$ , i.e.  $\sup_j C_{j,h} = C_h < \infty$ , resp.  $\sup_j C_{j,m} = C_m < \infty$ . Then, there exist  $p_j \in \mathcal{C}^\infty(\mathbb{R}^{2d})$  such that

$p_j \sim \sum_k q_k^{(j)}$ , for all  $j \in \mathbb{N}$  and  $\sum_j p_j \in FS_{A_p, A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ . Moreover,  $\sum_{j=0}^{\infty} p_j \sim \sum_{j=0}^{\infty} \sum_{l=0}^j q_j^{(l)}$  in  $FS_{A_p, A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ .

**Remark 3.1.**  $p_j \sim \sum_k q_k^{(j)}$  should be understood as equivalence of the sums  $\underbrace{0 + \dots + 0}_j + p_j + 0 + \dots$  and  $\sum_k q_k^{(j)}$ .

*Proof.* Let  $R \geq 2B$  and take  $p_j$  as in the remark after theorem 2.4, i.e.  $p_j = \sum_{k=j}^{\infty} (1 - \chi_k) q_k^{(j)}$ ,

for  $\chi_k$  constructed there. First, we consider the  $(M_p)$  case. We will prove that  $\sum_j p_j \in FS_{A_p, A_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}; B, m)$ , for sufficiently large  $R$ . Let  $h > 0$  be arbitrary but fixed. Obviously, without losing generality, we can assume that  $h \leq 1$ . For simplicity, denote  $C_h$  by  $C$ . Using the fact that  $1 - \chi_k(x, \xi) = 0$  for  $(x, \xi) \in Q_{Rm_k}$ , we have the estimate

$$\begin{aligned}
& \frac{|D_\xi^\alpha D_x^\beta p_j(x, \xi)| \langle (x, \xi) \rangle^{\rho|\alpha| + \rho|\beta| + 2\rho j} e^{-M(m|\xi|)} e^{-M(m|x|)}}{(8hH)^{|\alpha| + |\beta| + 2j} A_\alpha A_\beta A_j A_j} \\
& \leq \sum_{k=j}^{\infty} \sum_{\substack{\gamma \leq \alpha \\ \delta \leq \beta}} \binom{\alpha}{\gamma} \binom{\beta}{\delta} \left| D_\xi^{\alpha-\gamma} D_x^{\beta-\delta} q_k^{(j)}(x, \xi) \right| e^{-M(m|\xi|)} e^{-M(m|x|)} \\
& \quad \cdot \frac{|D_\xi^\gamma D_x^\delta (1 - \chi_k(x, \xi))| \langle (x, \xi) \rangle^{\rho|\alpha| + \rho|\beta| + 2\rho j}}{(8hH)^{|\alpha| + |\beta| + 2j} A_\alpha A_\beta A_j A_j} \\
& \leq C \sum_{k=j}^{\infty} \sum_{\substack{\gamma \leq \alpha \\ \delta \leq \beta}} \binom{\alpha}{\gamma} \binom{\beta}{\delta} \frac{h^{|\alpha| - |\gamma| + |\beta| - |\delta| + 2k} A_{\alpha-\gamma} A_{\beta-\delta} A_k A_k}{(8hH)^{|\alpha| + |\beta| + 2j} A_\alpha A_\beta A_j A_j} \\
& \quad \cdot \langle (x, \xi) \rangle^{\rho|\gamma| + \rho|\delta| + 2\rho j - 2\rho k} |D_\xi^\gamma D_x^\delta (1 - \chi_k(x, \xi))| \\
& \leq (c_0 c'_0)^2 C \sum_{k=j}^{\infty} \frac{1}{8^{|\alpha| + |\beta| + 2j} H^{2j}} h^{2(k-j)} H^{2k} L^{2(k-j)} M_{k-j}^{2\rho} |1 - \chi_k(x, \xi)| \langle (x, \xi) \rangle^{2\rho(j-k)} \\
& \quad + (c_0 c'_0)^2 C \sum_{k=j}^{\infty} \frac{1}{8^{|\alpha| + |\beta| + 2j} H^{2j}} \sum_{\substack{\gamma \leq \alpha, \delta \leq \beta \\ (\delta, \gamma) \neq (0,0)}} \binom{\alpha}{\gamma} \binom{\beta}{\delta} \\
& \quad \cdot \frac{h^{2(k-j)} H^{2k} L^{2(k-j)} M_{k-j}^{2\rho} |D_\xi^\gamma D_x^\delta (1 - \chi_k(x, \xi))| \langle (x, \xi) \rangle^{\rho|\gamma| + \rho|\delta| + 2\rho j - 2\rho k}}{h^{|\gamma| + |\delta|} A_\gamma A_\delta} \\
& = S_1 + S_2,
\end{aligned}$$

where  $S_1$  and  $S_2$  are the first and the second sum, correspondingly. To estimate  $S_1$  note



that, on the support of  $1 - \chi_k$ , the inequality  $\langle(x, \xi)\rangle \geq Rm_k$  holds. One obtains

$$S_1 \leq (c_0 c'_0)^2 C \sum_{k=j}^{\infty} \frac{(hLH)^{2(k-j)} M_{k-j}^{2\rho}}{R^{2\rho(k-j)} m_k^{2\rho(k-j)}} \leq (c_0 c'_0)^2 C \sum_{k=0}^{\infty} \frac{(hLH)^{2k}}{R^{2\rho k}} < \infty,$$

for  $R^\rho \geq 2LH \geq 2hLH$  (in the second inequality we use the fact that  $m_j^j \geq M_j$ ). For the estimate of  $S_2$ , note that  $D_\xi^\gamma D_x^\delta (1 - \chi_k(x, \xi)) = 0$  when  $(x, \xi) \in Q_{3Rm_k}^c$ , because  $(\delta, \gamma) \neq (0, 0)$  and  $\chi_k(x, \xi) = 0$  on  $Q_{3Rm_k}^c$ . So, for  $(x, \xi) \in Q_{3Rm_k}$ , we have that  $\langle(x, \xi)\rangle \leq \langle x \rangle + \langle \xi \rangle \leq 6Rm_k$ . Moreover, from the construction of  $\chi$ , we have that for the chosen  $h$ , there exists  $C_1 > 0$  such that  $|D_\xi^\alpha D_x^\beta \chi(x, \xi)| \leq C_1 h^{|\alpha|+|\beta|} A_\alpha A_\beta$ . By using  $m_k^k \geq M_k$ , one obtains

$$\begin{aligned} S_2 &\leq (c_0 c'_0)^2 C C_1 \sum_{k=j}^{\infty} \frac{1}{8^{|\alpha|+|\beta|+2j}} \sum_{\substack{\gamma \leq \alpha, \delta \leq \beta \\ (\delta, \gamma) \neq (0,0)}} \binom{\alpha}{\gamma} \binom{\beta}{\delta} \frac{(hLH)^{2(k-j)} 6^{\rho|\gamma|+\rho|\delta|} M_{k-j}^{2\rho} (Rm_k)^{\rho|\gamma|+\rho|\delta|}}{R^{2\rho(k-j)} m_k^{2\rho(k-j)} (Rm_k)^{|\gamma|+|\delta|}} \\ &\leq (c_0 c'_0)^2 C C_1 \sum_{k=0}^{\infty} \frac{(hLH)^{2k}}{R^{2\rho k}}, \end{aligned}$$

which is convergent for  $R^\rho \geq 2LH \geq 2hLH$ . Moreover, note that the choice of  $R$  for these sums to be convergent does not depend on  $j$ , hence  $\chi_k$  can be chosen to be the same for all  $p_j$ . So, these estimates does not depend on  $j$  and from this it follows that  $\sum_j p_j \in FS_{A_p, A_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d})$  (actually, to be precise,  $\sum_j p_j \in FS_{A_p, A_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}; B, m)$ , i.e. the same space as for  $\sum_k q_k^{(j)}$ ).

In the  $\{M_p\}$  case, there exist  $h_1, C_1 > 0$  such that  $|D_\xi^\alpha D_x^\beta \chi(x, \xi)| \leq C_1 h_1^{|\alpha|+|\beta|} A_\alpha A_\beta$ . Arguing in similar fashion, one proves that  $\sum_j p_j \in FS_{A_p, A_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{2d}; B, 8\tilde{h}H)$ , where  $\tilde{h} = \max\{h, h_1\}$ , i.e.  $\sum_j p_j \in FS_{A_p, A_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{2d})$ .

It remains to prove the second part of the lemma. One easily proves that  $\sum_{j=0}^{\infty} \sum_{l=0}^j q_j^{(l)} \in FS_{A_p, A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ . Note that  $\sum_{j=0}^{N-1} p_j - \sum_{j=0}^{N-1} \sum_{l=0}^j q_j^{(l)} = \sum_{j=0}^{N-1} \left( p_j - \sum_{k=j}^{N-1} q_k^{(j)} \right)$ . Moreover, for  $(x, \xi) \in Q_{3Rm_N}^c$  and  $N > j$ ,  $p_j - \sum_{k=j}^{N-1} q_k^{(j)} = \sum_{k=N}^{\infty} (1 - \chi_k) q_k^{(j)}$ . This easily follows from the definition of  $\chi_k$  and the fact that  $m_n$  is monotonically increasing. We will consider first the  $(M_p)$  case. For arbitrary but fixed  $0 < h \leq 1$  and  $(x, \xi) \in Q_{3Rm_N}^c$ , we estimate as follows

$$\begin{aligned} &\frac{\left| D_\xi^\alpha D_x^\beta \sum_{k=N}^{\infty} (1 - \chi_k(x, \xi)) q_k^{(j)}(x, \xi) \right| \langle(x, \xi)\rangle^{\rho|\alpha|+\rho|\beta|+2\rho N} e^{-M(m|\xi|)} e^{-M(m|x|)}}{(8(1+H)h)^{|\alpha|+|\beta|+2N} A_\alpha A_\beta A_N A_N} \\ &\leq \sum_{k=N}^{\infty} \frac{(1 - \chi_k(x, \xi)) \left| D_\xi^\alpha D_x^\beta q_k^{(j)}(x, \xi) \right| \langle(x, \xi)\rangle^{\rho|\alpha|+\rho|\beta|+2\rho N} e^{-M(m|\xi|)} e^{-M(m|x|)}}{(8(1+H)h)^{|\alpha|+|\beta|+2N} A_\alpha A_\beta A_N A_N} \end{aligned}$$

$$\begin{aligned}
& + \sum_{k=N}^{\infty} \sum_{\substack{\gamma \leq \alpha, \delta \leq \beta \\ (\delta, \gamma) \neq (0,0)}} \binom{\alpha}{\gamma} \binom{\beta}{\delta} \left| D_{\xi}^{\alpha-\gamma} D_x^{\beta-\delta} q_k^{(j)}(x, \xi) \right| e^{-M(m|\xi|)} e^{-M(m|x|)} \\
& \quad \cdot \frac{|D_{\xi}^{\gamma} D_x^{\delta} (1 - \chi_k(x, \xi))| \langle (x, \xi) \rangle^{\rho|\alpha| + \rho|\beta| + 2\rho N}}{(8(1+H)h)^{|\alpha| + |\beta| + 2N} A_{\alpha} A_{\beta} A_N A_N} \\
& \leq \frac{C}{64^N} \sum_{k=N}^{\infty} \frac{(1 - \chi_k(x, \xi)) h^{2k-2N} A_k A_k}{(1+H)^{2N} \langle (x, \xi) \rangle^{2\rho k - 2\rho N} A_N A_N} \\
& \quad + \frac{C}{64^N} \sum_{k=N}^{\infty} \frac{1}{8^{|\alpha| + |\beta|}} \sum_{\substack{\gamma \leq \alpha, \delta \leq \beta \\ (\delta, \gamma) \neq (0,0)}} \binom{\alpha}{\gamma} \binom{\beta}{\delta} \frac{h^{2k-2N} |D_{\xi}^{\gamma} D_x^{\delta} (1 - \chi_k(x, \xi))| \langle (x, \xi) \rangle^{\rho|\gamma| + \rho|\delta|} A_k A_k}{(1+H)^{2N} h^{|\gamma| + |\delta|} \langle (x, \xi) \rangle^{2\rho k - 2\rho N} A_{\gamma} A_{\delta} A_N A_N} \\
& = S_1 + S_2,
\end{aligned}$$

where  $S_1$  and  $S_2$  are the first and the second sum, correspondingly. To estimate  $S_1$ , observe that on the support of  $1 - \chi_k$  the inequality  $\langle (x, \xi) \rangle \geq Rm_k$  holds. Using the monotone increasingness of  $m_n$  and (M.2) for  $A_p$ , one obtains

$$\begin{aligned}
S_1 & \leq \frac{c_0^2 C}{64^N} \sum_{k=N}^{\infty} \frac{h^{2k-2N} H^{2k} A_{k-N} A_{k-N}}{(1+H)^{2N} R^{2\rho k - 2\rho N} m_k^{2\rho k - 2\rho N}} \leq \frac{(c_0 c'_0)^2 C}{64^N} \sum_{k=N}^{\infty} \frac{h^{2k-2N} H^{2k} L^{2k-2N} M_{k-N}^{2\rho}}{(1+H)^{2N} R^{2\rho k - 2\rho N} m_{k-N}^{2\rho k - 2\rho N}} \\
& = \frac{(c_0 c'_0)^2 C}{64^N} \frac{H^{2N}}{(1+H)^{2N}} \sum_{k=0}^{\infty} \left( \frac{hHL}{R^{\rho}} \right)^{2k} \leq \frac{(c_0 c'_0)^2 C}{64^N} \sum_{k=0}^{\infty} \left( \frac{HL}{R^{\rho}} \right)^{2k} = \frac{(c_0 c'_0)^2 C \tilde{C}}{64^N},
\end{aligned}$$

where we put  $\tilde{C} = \sum_{k=0}^{\infty} \left( \frac{HL}{R^{\rho}} \right)^{2k}$ , for some fixed  $R^{\rho} \geq 2HL$ . For the sum  $S_2$ , observe that

$D_{\xi}^{\gamma} D_x^{\delta} (1 - \chi_k(x, \xi)) = 0$  when  $(x, \xi) \in Q_{3Rm_k}^c$ , because  $(\delta, \gamma) \neq (0, 0)$  and  $\chi_k(x, \xi) = 0$  on  $Q_{3Rm_k}^c$ . Moreover, from the construction of  $\chi$ , we have that for the chosen  $h$ , there exists  $C_1 > 1$  such that  $|D_{\xi}^{\alpha} D_x^{\beta} \chi(x, \xi)| \leq C_1 h^{|\alpha| + |\beta|} A_{\alpha} B_{\beta}$ . Now

$$\begin{aligned}
S_2 & \leq \frac{c_0^2 C C_1}{64^N} \sum_{k=N}^{\infty} \frac{1}{8^{|\alpha| + |\beta|}} \sum_{\substack{\gamma \leq \alpha, \delta \leq \beta \\ (\delta, \gamma) \neq (0,0)}} \binom{\alpha}{\gamma} \binom{\beta}{\delta} \frac{h^{2k-2N} 6^{|\gamma| + |\delta|} H^{2k} A_{k-N} A_{k-N}}{(1+H)^{2N} R^{2\rho k - 2\rho N} m_k^{2\rho k - 2\rho N}} \\
& \leq \frac{c_0^2 C C_1}{64^N} \sum_{k=N}^{\infty} \frac{h^{2k-2N} H^{2k} A_{k-N} A_{k-N}}{(1+H)^{2N} R^{2\rho k - 2\rho N} m_k^{2\rho k - 2\rho N}} \leq \frac{(c_0 c'_0)^2 C C_1 \tilde{C}}{64^N},
\end{aligned}$$

where we used the above estimate for the last sum. So, we have

$$\begin{aligned}
& \frac{\left| D_{\xi}^{\alpha} D_x^{\beta} \sum_{j=0}^{N-1} \left( p_j(x, \xi) - \sum_{k=j}^{N-1} q_k^{(j)}(x, \xi) \right) \right| \langle (x, \xi) \rangle^{\rho|\alpha| + \rho|\beta| + 2\rho N} e^{-M(m|\xi|)} e^{-M(m|x|)}}{(8(1+H)h)^{|\alpha| + |\beta| + 2N} A_{\alpha} A_{\beta} A_N A_N} \\
& \leq \sum_{j=0}^{N-1} \frac{\left| D_{\xi}^{\alpha} D_x^{\beta} \left( p_j(x, \xi) - \sum_{k=j}^{N-1} q_k^{(j)}(x, \xi) \right) \right| \langle (x, \xi) \rangle^{\rho|\alpha| + \rho|\beta| + 2\rho N} e^{-M(m|\xi|)} e^{-M(m|x|)}}{(8(1+H)h)^{|\alpha| + |\beta| + 2N} A_{\alpha} A_{\beta} A_N A_N}
\end{aligned}$$

$$\leq \sum_{j=0}^{N-1} \frac{2(c_0 c'_0)^2 C C_1 \tilde{C}}{64^N} = \frac{2N(c_0 c'_0)^2 C C_1 \tilde{C}}{64^N},$$

which is bounded uniformly for all  $N \in \mathbb{Z}_+$ , for  $(x, \xi) \in Q_{3Rm_N}^c$ ,  $\alpha, \beta \in \mathbb{N}^d$ . The proof for the  $\{M_p\}$  case is similar.  $\square$

**Theorem 3.2.** *Let  $b \in \Gamma_{A_p, A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ . There exist  $a \in \Gamma_{A_p, A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$  and  $*$ -regularizing operator  $T$  such that  $b^w = A_a + T$ .*

*Proof.* Put  $p'_{0,0} = b$  and  $p'_{k,0} = 0$  for all  $k \in \mathbb{Z}_+$ . For  $j \in \mathbb{Z}_+$ , define  $p'_{0,j} = \dots = p'_{j-1,j} = 0$  and

$$p'_{k,j}(x, \xi) = \sum_{\substack{l_1+l_2+\dots+l_j=k \\ l_1 \geq 1, \dots, l_j \geq 1}} \sum_{|\alpha^{(1)}+\beta^{(1)}|=2l_1, \dots, |\alpha^{(j)}+\beta^{(j)}|=2l_j} \frac{c_{\alpha^{(1)}, \beta^{(1)}} \cdot \dots \cdot c_{\alpha^{(j)}, \beta^{(j)}}}{\alpha^{(1)}! \beta^{(1)}! \cdot \dots \cdot \alpha^{(j)}! \beta^{(j)}!} \cdot \partial_{\xi}^{\alpha^{(1)}+\dots+\alpha^{(j)}} \partial_x^{\beta^{(1)}+\dots+\beta^{(j)}} b(x, \xi),$$

for  $k \geq j$ ,  $k \in \mathbb{Z}_+$ . We will prove that  $\sum_k p'_{k,j}$  is an element of  $FS_{A_p, A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ . To do this note that, for  $k \geq j$ ,

$$\begin{aligned} & \left| \partial_{\xi}^{\gamma+\alpha^{(1)}+\dots+\alpha^{(j)}} \partial_x^{\delta+\beta^{(1)}+\dots+\beta^{(j)}} b(x, \xi) \right| \\ & \leq c_0^2 \|b\|_{h,m,\Gamma} \frac{h^{|\gamma|+|\delta|+2k} H^{|\gamma|+|\delta|+2k} A_{\gamma} A_{\delta} A_{2k} e^{M(m|\xi|)} e^{M(m|x|)}}{\langle (x, \xi) \rangle^{\rho|\gamma|+\rho|\delta|+2\rho k}} \\ & \leq c_0^3 \|b\|_{h,m,\Gamma} \frac{(hH^2)^{|\gamma|+|\delta|+2k} A_{\gamma} A_{\delta} A_k A_k e^{M(m|\xi|)} e^{M(m|x|)}}{\langle (x, \xi) \rangle^{\rho|\gamma|+\rho|\delta|+2\rho k}}. \end{aligned}$$

If we use the same estimates as in the beginning of the proof of theorem 3.1, we have

$$\frac{|c_{\alpha^{(s)}, \beta^{(s)}}|}{\alpha^{(s)}! \beta^{(s)}!} \leq \frac{c' d^{2l_s}}{\sqrt{|\alpha^{(s)}|! |\beta^{(s)}|!}} \leq c' d^{2l_s}, \quad (7)$$

for all  $s \in \{1, \dots, j\}$ , where  $c' = \frac{1}{\pi^d} \int_{\mathbb{R}^{2d}} e^{-|y|^2/2 - |\eta|^2/2} dy d\eta$ . Hence

$$\frac{|c_{\alpha^{(1)}, \beta^{(1)}}| \cdot \dots \cdot |c_{\alpha^{(j)}, \beta^{(j)}}|}{\alpha^{(1)}! \beta^{(1)}! \cdot \dots \cdot \alpha^{(j)}! \beta^{(j)}!} \leq c'^j d^{2k} \leq (c' d^2)^k.$$

The number of ways we can choose the positive integers  $l_1, \dots, l_j$  such that  $l_1 + \dots + l_j = k$  is  $\binom{k-1}{j-1}$ . For every fixed  $l_1, \dots, l_j$ , we have

$$\sum_{|\alpha^{(s)}+\beta^{(s)}|=2l_s} 1 = \binom{2l_s+2d-1}{2d-1} \leq 2^{2l_s+2d-1} = 2^{2d-1} 4^{l_s},$$

for  $s \in \{1, \dots, j\}$ . So, if we use that  $k \geq j$ , we have

$$\sum_{\substack{l_1+l_2+\dots+l_j=k \\ l_1 \geq 1, \dots, l_j \geq 1}} \sum_{|\alpha^{(1)}+\beta^{(1)}|=2l_1, \dots, |\alpha^{(j)}+\beta^{(j)}|=2l_j} 1 \leq 2^{j(2d-1)} 4^k \binom{k-1}{j-1} \leq 2^{k(2d-1)} 4^k 2^{k-1} \leq 2^{k(2d+2)}.$$

We obtain

$$\sum_{\substack{l_1+l_2+\dots+l_j=k \\ l_1 \geq 1, \dots, l_j \geq 1}} \sum_{|\alpha^{(1)}+\beta^{(1)}|=2l_1, \dots, |\alpha^{(j)}+\beta^{(j)}|=2l_j} \frac{|c_{\alpha^{(1)}, \beta^{(1)}}| \cdot \dots \cdot |c_{\alpha^{(j)}, \beta^{(j)}}|}{\alpha^{(1)}! \beta^{(1)}! \cdot \dots \cdot \alpha^{(j)}! \beta^{(j)}!} \leq (c' 2^{2d+2} d^2)^k,$$

i.e.

$$\frac{|D_\xi^\gamma D_x^\delta p'_{k,j}(x, \xi)| \langle (x, \xi) \rangle^{\rho|\gamma|+\rho|\delta|+2\rho k} e^{-M(m|\xi|)} e^{-M(m|x|)}}{(c' 2^{2d+2} d^2 h H^2)^{|\gamma|+|\delta|+2k} A_\gamma A_\delta A_k A_k} \leq c_0^3 \|b\|_{h,m,\Gamma},$$

for all  $(x, \xi) \in \mathbb{R}^{2d}$ ,  $\gamma, \delta \in \mathbb{N}^d$ ,  $k \in \mathbb{N}$  (for  $k < j$ ,  $p'_{k,j} = 0$ ). So  $\sum_k p'_{k,j} \in FS_{A_p, A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ . Note that  $c_0^3 \|b\|_{h,m}$  does not depend on  $j$ , i.e. the estimates are uniform in  $j$ . By the above lemma, there exist  $\mathcal{C}^\infty$  functions  $b_j$  such that  $b_j \sim \sum_k p'_{k,j}$ , for  $j \in \mathbb{N}$  and  $\sum_j b_j \in FS_{A_p, A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ . Note that, by the construction in the lemma and the way we define  $p'_{k,j}$ ,  $b_0 = p'_{0,0} = b$ . By theorem 2.4, there exists  $a \in \Gamma_{A_p, A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$  such that  $a \sim \sum_j (-1)^j b_j$ . We will prove that this  $a$  satisfies the conditions in the theorem. By theorem 3.1, there exist  $c \in \Gamma_{A_p, A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$  and \*-regularizing operator  $T_1$  such that  $A_a = c^w + T_1$  and  $c \sim \sum_{j=0}^{\infty} \sum_{|\alpha+\beta|=2j} \frac{c_{\alpha,\beta}}{\alpha! \beta!} \partial_\xi^\alpha \partial_x^\beta a(x, \xi)$ . One obtains

$$c \sim \sum_{j=0}^{\infty} \sum_{l+k=j} \sum_{|\alpha+\beta|=2l} (-1)^k \frac{c_{\alpha,\beta}}{\alpha! \beta!} \partial_\xi^\alpha \partial_x^\beta b_k(x, \xi).$$

To prove this, first, by using  $\sum_j (-1)^j b_j \in FS_{A_p, A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$  and (7), one easily verifies that the sum is an element of  $FS_{A_p, A_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ . Note that

$$\begin{aligned} & \sum_{j=0}^{N-1} \sum_{|\alpha+\beta|=2j} \frac{c_{\alpha,\beta}}{\alpha! \beta!} \partial_\xi^\alpha \partial_x^\beta a(x, \xi) - \sum_{j=0}^{N-1} \sum_{l=0}^j \sum_{|\alpha+\beta|=2l} (-1)^{j-l} \frac{c_{\alpha,\beta}}{\alpha! \beta!} \partial_\xi^\alpha \partial_x^\beta b_{j-l}(x, \xi) \\ &= \sum_{j=0}^{N-1} \sum_{|\alpha+\beta|=2j} \frac{c_{\alpha,\beta}}{\alpha! \beta!} \partial_\xi^\alpha \partial_x^\beta a(x, \xi) - \sum_{l=0}^{N-1} \sum_{j=l}^{N-1} \sum_{|\alpha+\beta|=2l} (-1)^{j-l} \frac{c_{\alpha,\beta}}{\alpha! \beta!} \partial_\xi^\alpha \partial_x^\beta b_{j-l}(x, \xi) \\ &= \sum_{j=0}^{N-1} \sum_{|\alpha+\beta|=2j} \frac{c_{\alpha,\beta}}{\alpha! \beta!} \partial_\xi^\alpha \partial_x^\beta a(x, \xi) - \sum_{j=0}^{N-1} \sum_{l=j}^{N-1} \sum_{|\alpha+\beta|=2j} (-1)^{l-j} \frac{c_{\alpha,\beta}}{\alpha! \beta!} \partial_\xi^\alpha \partial_x^\beta b_{l-j}(x, \xi) \\ &= \sum_{j=0}^{N-1} \sum_{|\alpha+\beta|=2j} \frac{c_{\alpha,\beta}}{\alpha! \beta!} \partial_\xi^\alpha \partial_x^\beta \left( a(x, \xi) - \sum_{s=0}^{N-j-1} (-1)^s b_s(x, \xi) \right). \end{aligned}$$

By using that  $a \sim \sum_j (-1)^j b_j$  and the inequality (7), one easily proves the desired equivalence. Now, observe that, if we prove the equivalence

$$b \sim \sum_{j=0}^{\infty} \sum_{l+k=j} \sum_{|\alpha+\beta|=2l} (-1)^k \frac{c_{\alpha,\beta}}{\alpha! \beta!} \partial_{\xi}^{\alpha} \partial_x^{\beta} b_k(x, \xi),$$

the claim of the theorem will follow. Observe that

$$\begin{aligned} \sum_{j=0}^{N-1} \sum_{l+k=j} \sum_{|\alpha+\beta|=2l} (-1)^k \frac{c_{\alpha,\beta}}{\alpha! \beta!} \partial_{\xi}^{\alpha} \partial_x^{\beta} b_k(x, \xi) - b(x, \xi) \\ = \sum_{j=1}^{N-1} \sum_{l+k=j} \sum_{|\alpha+\beta|=2l} (-1)^k \frac{c_{\alpha,\beta}}{\alpha! \beta!} \partial_{\xi}^{\alpha} \partial_x^{\beta} b_k(x, \xi) \end{aligned} \quad (8)$$

$$= \sum_{k=1}^{N-1} (-1)^{k-1} \left( \sum_{j=k}^{N-1} \sum_{|\alpha+\beta|=2(j-k+1)} \frac{c_{\alpha,\beta}}{\alpha! \beta!} \partial_{\xi}^{\alpha} \partial_x^{\beta} b_{k-1}(x, \xi) - b_k(x, \xi) \right). \quad (9)$$

Because of the way we defined  $p'_{s,k}$ , for  $s \geq k \geq 2$ , we have

$$\begin{aligned} p'_{s,k}(x, \xi) &= \sum_{l=1}^{s-k+1} \sum_{|\alpha+\beta|=2l} \frac{c_{\alpha,\beta}}{\alpha! \beta!} \partial_{\xi}^{\alpha} \partial_x^{\beta} \sum_{\substack{l_1+\dots+l_{k-1}=s-l \\ l_1 \geq 1, \dots, l_{k-1} \geq 1}} \sum_{|\alpha^{(1)}+\dots+\alpha^{(k-1)}|=2l_1, \dots, |\alpha^{(k-1)}+\beta^{(k-1)}|=2l_{k-1}} \\ &\quad \frac{c_{\alpha^{(1)},\beta^{(1)}} \cdot \dots \cdot c_{\alpha^{(k-1)},\beta^{(k-1)}}}{\alpha^{(1)}! \beta^{(1)}! \cdot \dots \cdot \alpha^{(k-1)}! \beta^{(k-1)}!} \partial_{\xi}^{\alpha^{(1)}+\dots+\alpha^{(k-1)}} \partial_x^{\beta^{(1)}+\dots+\beta^{(k-1)}} b(x, \xi) \\ &= \sum_{l=1}^{s-k+1} \sum_{|\alpha+\beta|=2l} \frac{c_{\alpha,\beta}}{\alpha! \beta!} \partial_{\xi}^{\alpha} \partial_x^{\beta} p'_{s-l,k-1}(x, \xi). \end{aligned}$$

For  $k = 1$  one easily checks that the same formula holds for  $p'_{s,1}$  (by definition,  $p'_{s-l,0} = 0$  when  $s > l$  and  $p'_{0,0} = b$ ). Hence

$$\begin{aligned} \sum_{s=k}^{N-1} p'_{s,k}(x, \xi) &= \sum_{l=1}^{N-k} \sum_{|\alpha+\beta|=2l} \frac{c_{\alpha,\beta}}{\alpha! \beta!} \partial_{\xi}^{\alpha} \partial_x^{\beta} \left( \sum_{s=l+k-1}^{N-1} p'_{s-l,k-1}(x, \xi) \right) \\ &= \sum_{l=1}^{N-k} \sum_{|\alpha+\beta|=2l} \frac{c_{\alpha,\beta}}{\alpha! \beta!} \partial_{\xi}^{\alpha} \partial_x^{\beta} \left( \sum_{s=k-1}^{N-l-1} p'_{s,k-1}(x, \xi) \right). \end{aligned}$$

Now, we obtain

$$\begin{aligned} \sum_{j=k}^{N-1} \sum_{|\alpha+\beta|=2(j-k+1)} \frac{c_{\alpha,\beta}}{\alpha! \beta!} \partial_{\xi}^{\alpha} \partial_x^{\beta} b_{k-1}(x, \xi) - b_k(x, \xi) \\ = \sum_{l=1}^{N-k} \sum_{|\alpha+\beta|=2l} \frac{c_{\alpha,\beta}}{\alpha! \beta!} \partial_{\xi}^{\alpha} \partial_x^{\beta} b_{k-1}(x, \xi) - \sum_{s=k}^{N-1} p'_{s,k}(x, \xi) + \sum_{s=k}^{N-1} p'_{s,k}(x, \xi) - b_k(x, \xi) \end{aligned}$$

$$= \sum_{l=1}^{N-k} \sum_{|\alpha+\beta|=2l} \frac{c_{\alpha,\beta}}{\alpha!\beta!} \partial_\xi^\alpha \partial_x^\beta \left( b_{k-1}(x, \xi) - \sum_{s=k-1}^{N-l-1} p'_{s,k-1}(x, \xi) \right) + \sum_{s=k}^{N-1} p'_{s,k}(x, \xi) - b_k(x, \xi).$$

By construction,  $b_{k-1} \sim \underbrace{0 + \dots + 0}_{k-1} + \sum_{s=k-1}^{\infty} p'_{s,k-1}$ . Moreover, by the above estimates for the derivatives of  $p'_{s,k}$ , the above lemma and its prove it follows that there exist  $B > 0$ ,  $m > 0$  and  $\tilde{C}_h > 0$  in the  $(M_p)$  case, resp. there exist  $B > 0$ ,  $h > 0$  and  $\tilde{C}_m > 0$  in the  $\{M_p\}$  case, such that for every  $h > 0$

$$\frac{|D_\xi^\alpha D_x^\beta (b_k(x, \xi) - \sum_{s < N} p'_{s,k}(x, \xi))| \langle (x, \xi) \rangle^{\rho|\alpha| + \rho|\beta| + 2N\rho} e^{-M(m|\xi|)} e^{-M(m|x|)}}{h^{|\alpha| + |\beta| + 2N} A_\alpha A_\beta A_N A_N} \leq \tilde{C}_h,$$

for all  $(x, \xi) \in Q_{Bm_N}^c$ ,  $\alpha, \beta \in \mathbb{N}^d$  and  $k, N \in \mathbb{N}$ ,  $N > k$ , in the  $(M_p)$  case, resp. the same as above but for some  $h$  and every  $m$  with  $\tilde{C}_m$  in place of  $\tilde{C}_h$ , in the  $\{M_p\}$  case. Now, if we use the estimate (7), we get that

$$\begin{aligned} & \left| \sum_{|\alpha+\beta|=2l} \frac{c_{\alpha,\beta}}{\alpha!\beta!} \partial_\xi^{\alpha+\gamma} \partial_x^{\beta+\delta} \left( b_{k-1}(x, \xi) - \sum_{s=k-1}^{N-l-1} p'_{s,k-1}(x, \xi) \right) \right| \\ & \leq \tilde{C} \sum_{|\alpha+\beta|=2l} \frac{|c_{\alpha,\beta}|}{\alpha!\beta!} \frac{h^{|\gamma| + |\delta| + 2N} A_{\alpha+\gamma} A_{\beta+\delta} A_{N-l} A_{N-l} e^{M(m|\xi|)} e^{M(m|x|)}}{\langle (x, \xi) \rangle^{\rho|\gamma| + \rho|\delta| + 2N\rho}} \\ & \leq c_0^3 \tilde{C}' d^{2l} \sum_{|\alpha+\beta|=2l} \frac{(hH^2)^{|\gamma| + |\delta| + 2N} A_\gamma A_\delta A_N A_N e^{M(m|\xi|)} e^{M(m|x|)}}{\langle (x, \xi) \rangle^{\rho|\gamma| + \rho|\delta| + 2N\rho}} \\ & \leq c_0^3 \tilde{C}' 2^{2d-1} \frac{(2hdH^2)^{|\gamma| + |\delta| + 2N} A_\gamma A_\delta A_N A_N e^{M(m|\xi|)} e^{M(m|x|)}}{\langle (x, \xi) \rangle^{\rho|\gamma| + \rho|\delta| + 2N\rho}}, \end{aligned}$$

for all  $(x, \xi) \in Q_{Bm_N}^c$ ,  $\gamma, \delta \in \mathbb{N}^d$ ,  $N \geq l + 1$  (in the last inequality we used  $\sum_{|\alpha+\beta|=2l} 1 \leq$

$2^{2l+2d-1}$ ), where we put  $\tilde{C} = \tilde{C}_h$  in the  $(M_p)$  case, resp.  $\tilde{C} = \tilde{C}_m$  in the  $\{M_p\}$  case. Note that the estimates are uniform in  $l$  and  $k$ . One obtains

$$\begin{aligned} & \left| \partial_\xi^\gamma \partial_x^\delta \left( \sum_{l=1}^{N-k} \sum_{|\alpha+\beta|=2l} \frac{c_{\alpha,\beta}}{\alpha!\beta!} \partial_\xi^\alpha \partial_x^\beta \left( b_{k-1}(x, \xi) - \sum_{s=k-1}^{N-l-1} p'_{s,k-1}(x, \xi) \right) \right) \right| \\ & \leq c_0^3 \tilde{C}' 2^{2d-1} \frac{(4hdH^2)^{|\gamma| + |\delta| + 2N} A_\gamma A_\delta A_N A_N e^{M(m|\xi|)} e^{M(m|x|)}}{\langle (x, \xi) \rangle^{\rho|\gamma| + \rho|\delta| + 2N\rho}}, \end{aligned}$$

for all  $(x, \xi) \in Q_{Bm_N}^c$ ,  $\gamma, \delta \in \mathbb{N}^d$ ,  $N > k$ , with uniform estimates in  $k$ . Similar estimates hold for  $\sum_{s=k}^{N-1} p'_{s,k}(x, \xi) - b_k(x, \xi)$  (by the definition of  $b_k$ ). By using the equality (9), we obtain the desired result.  $\square$

The importance in the study of the Anti-Wick quantization lies in the following results. The proofs are similar to the case of Schwartz distributions and we omit them (see for example [10]).

**Proposition 3.5.** *Let  $a$  be a locally integrable function with  $*$ -ultrapolynomial growth (for example, an element of  $\Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ ). If  $a(x, \xi) \geq 0$  for almost every  $(x, \xi) \in \mathbb{R}^{2d}$ , then  $(A_a u, u)_{L^2} \geq 0$ ,  $\forall u \in \mathcal{S}^*$ . Moreover, if  $a(x, \xi) > 0$  for almost every  $(x, \xi) \in \mathbb{R}^{2d}$ , then  $(A_a u, u)_{L^2} > 0$ ,  $\forall u \in \mathcal{S}^*$ ,  $u \neq 0$ .*

Nontrivial symbols  $a$  that satisfy the conditions of this proposition, for example, are the ultrapolynomials of the form  $\sum_{\alpha} c_{2\alpha} \xi^{2\alpha}$ , where  $c_{2\alpha} > 0$  satisfy the necessary conditions for this to be an ultrapolynomial, i.e. there exist  $C > 0$  and  $\tilde{L} > 0$ , resp. for every  $\tilde{L} > 0$  there exists  $C > 0$ , such that  $|c_{2\alpha}| \leq C \tilde{L}^{2|\alpha|} / M_{2\alpha}$ , for all  $\alpha \in \mathbb{N}^d$ .

**Proposition 3.6.** *Let  $a \in L^\infty(\mathbb{R}^{2d})$ . Then  $A_a$  extends to a bounded operator on  $L^2$ , with the following estimate of its norm  $\|A_a\|_{\mathcal{L}_b(L^2(\mathbb{R}^d))} \leq \|a\|_{L^\infty(\mathbb{R}^{2d})}$ .*

## 4 Convolution with the gaussian kernel

The existence of the convolution of two ultradistributions was studied in [12] and [5] in the Beurling case and in [14] in the Roumieu case. The convolution of two ultradistributions  $S, T \in \mathcal{D}'^*$  exists if for every  $\varphi \in \mathcal{D}^*$ ,  $(S \otimes T)\varphi^\Delta \in \mathcal{D}'_{L^1}^{(M_p)}(\mathbb{R}^{2d})$ , resp.  $(S \otimes T)\varphi^\Delta \in \tilde{\mathcal{D}}'_{L^1}^{\{M_p\}}(\mathbb{R}^{2d})$ , where  $\varphi^\Delta(x, y) = \varphi(x + y)$ . In that case  $S * T$  is defined by  $\langle S * T, \varphi \rangle = \langle (S \otimes T)\varphi^\Delta, 1 \rangle$ . We will briefly comment on the meaning of  $\langle (S \otimes T)\varphi^\Delta, 1 \rangle$  (for the complete theory of the existence of convolution as well as other equivalent definitions, we refer to [12] and [5] for the Beurling case and [14] for the Roumieu case). In [12], for the Beurling case and [14] for the Roumieu case, alternative Hausdorff locally convex topology is introduced on  $\mathcal{D}_{L^\infty}^{(M_p)}$ , resp.  $\tilde{\mathcal{D}}_{L^\infty}^{\{M_p\}}$ , which is weaker than the original topology, stronger than the induced one from  $\mathcal{E}^*$  and  $\mathcal{D}^*$  is continuously and densely injected in it. Moreover, the duals of these spaces with these topologies coincide with  $\mathcal{D}'_{L^1}^{(M_p)}$ , resp. with  $\tilde{\mathcal{D}}'_{L^1}^{\{M_p\}}$  as sets. The meaning of  $\langle (S \otimes T)\varphi^\Delta, 1 \rangle$  is in the sense of these dualities. If  $\psi \in \mathcal{D}^*$  is such that  $0 \leq \psi \leq 1$ ,  $\psi(x) = 1$  when  $|x| \leq 1$  and  $\psi(x) = 0$  when  $|x| > 2$ , then  $\psi_j \rightarrow 1$ , when  $j \rightarrow \infty$ , in the alternative topology of  $\mathcal{D}_{L^\infty}^{(M_p)}$ , resp.  $\tilde{\mathcal{D}}_{L^\infty}^{\{M_p\}}$ . So, if  $G \in \mathcal{D}'_{L^1}^{(M_p)}$ , resp.  $G \in \tilde{\mathcal{D}}'_{L^1}^{\{M_p\}}$ ,  $\langle G, \psi_j \rangle \rightarrow \langle G, 1 \rangle$ , when  $j \rightarrow \infty$ .

Our goal in this section is to find the largest subspace of  $\mathcal{D}'^*$  such that the convolution of each element of that subspace with  $e^{s|\cdot|^2}$  exists, where  $s \in \mathbb{R}$ ,  $s \neq 0$  is fixed. The general idea is similar to that in [21], where the case of Schwartz distributions is considered. We will need the following results, concerning the Laplace transform, from [15].

For a set  $B \subseteq \mathbb{R}^d$  denote by  $\text{ch } B$  the convex hull of  $B$ .

**Theorem 4.1.** *Let  $B$  be a connected open set in  $\mathbb{R}_\xi^d$  and  $T \in \mathcal{D}'^*(\mathbb{R}_x^d)$  be such that, for all  $\xi \in B$ ,  $e^{-x\xi}T(x) \in \mathcal{S}'^*(\mathbb{R}_x^d)$ . Then the Fourier transform  $\mathcal{F}_{x \rightarrow \eta}(e^{-x\xi}T(x))$  is an analytic function of  $\zeta = \xi + i\eta$  for  $\xi \in \text{ch } B$ ,  $\eta \in \mathbb{R}^d$ . Furthermore, it satisfies the following*



estimates: for every  $K \subset\subset \text{ch } B$  there exist  $k > 0$  and  $C > 0$ , resp. for every  $k > 0$  there exists  $C > 0$ , such that

$$|\mathcal{F}_{x \rightarrow \eta}(e^{-x\xi}T(x))(\xi + i\eta)| \leq Ce^{M(k|\eta|)}, \forall \xi \in K, \forall \eta \in \mathbb{R}^d. \quad (10)$$

**Remark 4.1.** If, for  $S \in \mathcal{D}'^*$ , the conditions of the theorem are fulfilled, we call  $\mathcal{F}_{x \rightarrow \eta}(e^{-x\xi}S(x))$  the Laplace transform of  $S$  and denote it by  $\mathcal{L}(S)$ . Moreover,

$$\mathcal{L}(S)(\zeta) = \left\langle e^{\varepsilon\sqrt{1+|x|^2}}e^{-x\zeta}S(x), e^{-\varepsilon\sqrt{1+|x|^2}} \right\rangle, \quad (11)$$

for  $\zeta \in U + i\mathbb{R}_\eta^d$ , where  $\overline{U} \subset\subset \text{ch } B$  and  $\varepsilon$  depends on  $U$ .

If for  $S \in \mathcal{D}'^*$  the conditions of the theorem are fulfilled for  $B = \mathbb{R}^d$ , then the choice of  $\varepsilon$  can be made uniform for all  $K \subset\subset \mathbb{R}^d$ .

**Theorem 4.2.** Let  $B$  be a connected open set in  $\mathbb{R}_\xi^d$  and  $f$  an analytic function on  $B + i\mathbb{R}_\eta^d$ . Let  $f$  satisfies the condition: for every compact subset  $K$  of  $B$  there exist  $C > 0$  and  $k > 0$ , resp. for every  $k > 0$  there exists  $C > 0$ , such that

$$|f(\xi + i\eta)| \leq Ce^{M(k|\eta|)}, \forall \xi \in K, \forall \eta \in \mathbb{R}^d. \quad (12)$$

Then, there exists  $S \in \mathcal{D}'^*(\mathbb{R}_x^d)$  such that  $e^{-x\xi}S(x) \in \mathcal{S}'^*(\mathbb{R}_x^d)$ , for all  $\xi \in B$  and

$$\mathcal{L}(S)(\xi + i\eta) = \mathcal{F}_{x \rightarrow \eta}(e^{-x\xi}S(x))(\xi + i\eta) = f(\xi + i\eta), \quad \xi \in B, \eta \in \mathbb{R}^d. \quad (13)$$

Put  $B^* = \{S \in \mathcal{D}'^* | \cosh(k|x|)S \in \mathcal{S}'^*, \forall k \geq 0\}$  and for  $s \in \mathbb{R} \setminus \{0\}$ , put  $B_s^* = e^{-s|x|^2}B^*$ . Obviously  $B^* \subseteq \mathcal{S}'^*$  and  $B_s^* \subseteq \mathcal{D}'^*$ . Define

$$A^* = \{f \in \mathcal{O}(\mathbb{C}^d) | \forall K \subset\subset \mathbb{R}_\xi^d, \exists h, C > 0, \text{ resp. } \forall h > 0, \exists C > 0, \text{ such that } |f(\xi + i\eta)| \leq Ce^{M(h|\eta|)}, \forall \xi \in K, \forall \eta \in \mathbb{R}^d\},$$

$A_{\text{real}}^* = \{f|_{\mathbb{R}^d} | f \in A^*\}$  and  $A_s^* = e^{s|x|^2}A_{\text{real}}^*$ . Assume that  $k > 0$ . First we will prove that  $\cosh(k|x|) \in \mathcal{C}^\infty(\mathbb{R}^d)$ . For  $\rho \geq 0$ , we have

$$\cosh(k\rho) = \frac{1}{2} \left( \sum_{n=0}^{\infty} \frac{k^n \rho^n}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n k^n \rho^n}{n!} \right) = \sum_{n=0}^{\infty} \frac{k^{2n} \rho^{2n}}{(2n)!},$$

hence  $\cosh(k|x|) = \sum_{n=0}^{\infty} \frac{k^{2n}|x|^{2n}}{(2n)!}$  and the function  $\sum_{n=0}^{\infty} \frac{k^{2n}|x|^{2n}}{(2n)!}$  is obviously in  $\mathcal{C}^\infty(\mathbb{R}^d)$ . We will give another two equivalent definitions of  $B^*$ . We need the following lemmas.

**Lemma 4.1.** Let  $k > 0$ . The function  $\frac{\cosh(k|x|)}{\cosh(2k|x|)}$  is an element of  $\mathcal{S}^*$ .

*Proof.* Consider the function  $g_k(z) = \sum_{n=0}^{\infty} \frac{k^{2n}(z^2)^n}{(2n)!}$ . Obviously  $g_k(z)$  is an entire function. Put  $W = \{z = x + iy \in \mathbb{C}^d \mid |x| > 2|y|\}$  and consider the set  $W_r = W \setminus \overline{B(0, r)}$ , where  $B(0, r)$  is the ball in  $\mathbb{C}^d$  with center at 0 and radius  $r > 0$ . Then  $\frac{e^{k\sqrt{z^2}} + e^{-k\sqrt{z^2}}}{2}$  is analytic and single valued function on  $W_r$ , where we take the principal branch of the square root which is analytic on  $\mathbb{C} \setminus (-\infty, 0]$ . Also, for  $z \in W_r$ , put  $\rho = \sqrt{(|x|^2 - |y|^2)^2 + 4(xy)^2}$ ,  $\cos \theta = \frac{|x|^2 - |y|^2}{\sqrt{(|x|^2 - |y|^2)^2 + 4(xy)^2}}$  and  $\sin \theta = \frac{2xy}{\sqrt{(|x|^2 - |y|^2)^2 + 4(xy)^2}}$ , where  $\theta \in (-\pi, \pi)$ , from what it follows  $\theta \in (-\pi/2, \pi/2)$  (because  $\cos \theta > 0$  and  $\theta \in (-\pi, \pi)$ ). We will need sharper estimate for  $\cos \theta$ .

$$\begin{aligned} \cos \theta &= \frac{|x|^2 - |y|^2}{\sqrt{(|x|^2 - |y|^2)^2 + 4(xy)^2}} = \left(1 + \left(\frac{2|xy|}{|x|^2 - |y|^2}\right)^2\right)^{-1/2} \\ &\geq \left(1 + \left(\frac{|x|^2 + |y|^2}{|x|^2 - |y|^2}\right)^2\right)^{-1/2} \geq \left(1 + \left(\frac{\frac{5}{4}|x|^2}{\frac{3}{4}|x|^2}\right)^2\right)^{-1/2} = \frac{3}{\sqrt{34}}. \end{aligned}$$

Then

$$\begin{aligned} \left|e^{k\sqrt{z^2}} + e^{-k\sqrt{z^2}}\right| &\geq \left|e^{k\sqrt{z^2}}\right| - \left|e^{-k\sqrt{z^2}}\right| = e^{k\operatorname{Re} \sqrt{\rho(\cos \theta + i \sin \theta)}} - e^{-k\operatorname{Re} \sqrt{\rho(\cos \theta + i \sin \theta)}} \\ &= e^{k\operatorname{Re} \sqrt{\rho}(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2})} - e^{-k\operatorname{Re} \sqrt{\rho}(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2})} \geq e^{k\sqrt{\rho} \cos \frac{\theta}{2}} - 1 \end{aligned}$$

where the second equality follows from the fact that we take the principal branch of the square root. Now, using the above estimate for  $\cos \theta$ , we have

$$\sqrt{\rho} \cos \frac{\theta}{2} = \sqrt{\rho} \sqrt{\frac{\cos \theta + 1}{2}} \geq \sqrt{\rho} \sqrt{\frac{3 + \sqrt{34}}{2\sqrt{34}}}.$$

So, if we put  $c_1 = \sqrt{\frac{3 + \sqrt{34}}{2\sqrt{34}}}$ , we obtain

$$\left|e^{k\sqrt{z^2}} + e^{-k\sqrt{z^2}}\right| \geq e^{c_1 k \sqrt{(|x|^2 - |y|^2)^2 + 4(xy)^2}} - 1 \geq e^{c_1 k \sqrt{|x|^2 - |y|^2}} - 1 > 0. \quad (14)$$

Hence  $e^{k\sqrt{z^2}} + e^{-k\sqrt{z^2}}$  doesn't have zeroes in  $W_r$ . Now,  $f(z) = \frac{e^{k\sqrt{z^2}} + e^{-k\sqrt{z^2}}}{e^{2k\sqrt{z^2}} + e^{-2k\sqrt{z^2}}}$  is an analytic function on  $W_r$ . Moreover, because  $\left(e^{k\sqrt{z^2}} + e^{-k\sqrt{z^2}}\right)/2 = g_k(z)$ , for  $z \in W_r \cap \mathbb{R}_x^d$  and from the uniqueness of analytic continuation, it follows  $\left(e^{k\sqrt{z^2}} + e^{-k\sqrt{z^2}}\right)/2 = g_k(z)$  on  $W_r$ . Hence  $f(z) = g_k(z)/g_{2k}(z)$  on  $W_r$  and this holds for all  $r > 0$ , hence on  $W$ . Note that

$g_{2k}(0) = 1$ , so, there exists  $r_0 > 0$  such that  $|g_{2k}(z)| > 0$  on  $B(0, 2r_0)$  and hence  $g_k(z)/g_{2k}(z)$  is analytic function on  $W \cup B(0, 2r_0)$ . Let  $C_{r_0} > 0$  be a constant such that  $|g_k(z)/g_{2k}(z)| \leq C_{r_0}$  on  $\overline{B(0, r_0)}$ . Take  $r_1 > 0$  such that  $\overline{B(x, 2dr_1)} \subseteq (\mathbb{C}^d \setminus \overline{B(0, r_0/16)}) \cap W$ , for all  $x \in W_{\frac{r_0}{4}} \cap \mathbb{R}_x^d$ . Then, for such  $x$ , from Cauchy integral formula, we have

$$|\partial_z^\alpha f(x)| \leq \frac{\alpha!}{r_1^{|\alpha|}} \sup_{|w_1-x_1| \leq r_1, \dots, |w_d-x_d| \leq r_1} |f(w)|.$$

Now, for  $w = u + iv \in \mathbb{C}^d$  such that  $|w_j - x_j| \leq r_1$ , for all  $j = 1, \dots, d$ , using the estimate (14) but with  $2k$  instead of  $k$  and the fact  $\operatorname{Re} \sqrt{z^2} > 0$ , for  $z \in W$ , which we proved above, we get

$$\begin{aligned} |f(w)| &= \left| \frac{e^{k\sqrt{w^2}} + e^{-k\sqrt{w^2}}}{e^{2k\sqrt{w^2}} + e^{-2k\sqrt{w^2}}} \right| \leq \frac{e^{k\sqrt{(|u|^2-|v|^2)^2+4(uv)^2}} + 1}{e^{2c_1k\sqrt{|u|^2-|v|^2}} - 1} \\ &\leq \frac{2e^{k\sqrt{|u|^2-|v|^2+2|uv|}}}{e^{2c_1k\sqrt{|u|^2-|v|^2}} - 1} \leq \frac{2e^{\sqrt{2}k|u|}}{e^{\sqrt{3}c_1k|u|} - 1} \leq C_1 e^{(\sqrt{2}-\sqrt{3}c_1)k|u|} \end{aligned}$$

and it is easy to check that  $\sqrt{2} - \sqrt{3}c_1 < 0$ . If we put  $c = \sqrt{3}c_1 - \sqrt{2}$ , we get

$$|f(w)| \leq C_1 e^{-ck|u|} \leq C_1 e^{-ck(|x|-|u-x|)} \leq C_1 e^{ckr_1\sqrt{d}} e^{-ck|x|} = C_2 e^{-ck|x|}.$$

Hence  $|\partial_x^\alpha f(x)| \leq C_2 \frac{\alpha!}{r_1^{|\alpha|}} e^{-ck|x|}$ . For  $x \in (B(0, r_0/2) \cap \mathbb{R}_x^d) \setminus \{0\}$ , if we take  $r_2 > 0$  small enough such that  $\overline{B(x, 2dr_2)} \subseteq B(0, r_0)$  we have (from Cauchy integral formula)

$$|\partial_x^\alpha f(x)| = \left| \partial_z^\alpha \left( \frac{g_k(x)}{g_{2k}(x)} \right) \right| \leq \frac{\alpha!}{r_2^{|\alpha|}} \sup_{|w_1-x_1| \leq r_2, \dots, |w_d-x_d| \leq r_2} |g(w)| \leq C_{r_0} \frac{\alpha!}{r_2^\alpha} \leq C_3 \frac{\alpha!}{r_2^\alpha} e^{-ck|x|}.$$

Because  $f(x)$  is in  $\mathcal{C}^\infty(\mathbb{R}^d)$  the same inequality will hold for the derivatives in  $x = 0$ . If we take  $r = \min\{r_1, r_2\}$  we get that, for  $x \in \mathbb{R}^d$ ,

$$|\partial_x^\alpha f(x)| \leq C \frac{\alpha!}{r^\alpha} e^{-ck|x|}, \quad (15)$$

for some  $C > 0$ . From this it easily follows that  $f(x) = \frac{\cosh(k|x|)}{\cosh(2k|x|)} \in \mathcal{S}^*$ .  $\square$

**Lemma 4.2.** *If  $\psi \in \mathcal{S}^*$  and  $T \in \mathcal{S}'^*$  then  $\psi T \in \mathcal{O}_C'^*$ .*

*Proof.* The Fourier transform is a bijection between  $\mathcal{O}_C^*$  and  $\mathcal{O}_M^*$  (see proposition 8 of [4]) and  $\mathcal{F}(\psi T) = \mathcal{F}\psi * \mathcal{F}T$ . Hence, it is enough to prove that  $\psi * T \in \mathcal{O}_M^*$  for all  $\psi \in \mathcal{S}^*$  and  $T \in \mathcal{S}'^*$ . From the representation theorem of ultradistributions in  $\mathcal{S}'^*$  (theorem 2 of [13]), there exists locally integrable function  $F(x)$  (in fact it can be taken to be

continuous) such that there exist  $m, C > 0$ , resp. for every  $m > 0$  there exists  $C > 0$ , such that  $\|F(x)e^{-M(m|x|)}\|_{L^\infty} \leq C$  and an ultradifferential operator  $P(D)$  of class  $*$  such that  $T = P(D)F$ . Because

$$\psi * T = \psi * P(D)F = P(D)(\psi * F) = P(D)\psi * F$$

and  $P(D)\psi \in \mathcal{S}^*$  it is enough to prove that for every  $\psi \in \mathcal{S}^*$  and every such  $F$ ,  $\psi * F \in \mathcal{O}_M^*$ . We will give the proof only in the  $\{M_p\}$  case, the  $(M_p)$  case is similar. Let  $\psi$  and  $F$  are such function. There exists  $h > 0$  such that

$$\sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{h^{|\alpha|} e^{M(h|x|)} |D^\alpha \psi(x)|}{M_\alpha} < \infty.$$

Take  $m$  such that  $\int_{\mathbb{R}^d} e^{-M(h|t|)} e^{M(2m|t|)} dt$  is finite. Later on we will impose another condition on  $m$ . Then  $\|F(x)e^{-M(m|x|)}\|_{L^\infty} \leq C_m$ . Note that  $e^{M(m|x-t|)} \leq 2e^{M(2m|x|)} e^{M(2m|t|)}$  (one easily proves that for  $\lambda, \nu > 0$ ,  $e^{M(\lambda+\nu)} \leq 2e^{M(2\lambda)} e^{M(2\nu)}$ ), so we have

$$\begin{aligned} |D^\alpha(\psi * F)(x)| &\leq \int_{\mathbb{R}^d} |D^\alpha \psi(t)| |F(x-t)| dt \leq C' C_m \int_{\mathbb{R}^d} \frac{e^{-M(h|t|)} M_\alpha}{h^{|\alpha|}} e^{M(m|x-t|)} dt \\ &\leq C' C_m C'' \frac{e^{M(2m|x|)} M_\alpha}{h^{|\alpha|}} \int_{\mathbb{R}^d} e^{-M(h|t|)} e^{M(2m|t|)} dt \leq C \frac{e^{M(2m|x|)} M_\alpha}{h^{|\alpha|}}. \end{aligned}$$

We will use the equivalent condition given in proposition 7 of [4] for a  $\mathcal{C}^\infty$  function to be a multiplier for  $\mathcal{S}'^{\{M_p\}}$ . Let  $k > 0$  be arbitrary but fixed. Take  $m$  small enough such that  $2m \leq k$ . Choose  $h_1 < h$ . Then, by the previous estimates, we obtain

$$\frac{h_1^{|\alpha|} e^{-M(k|x|)} |D^\alpha(\psi * F)(x)|}{M_\alpha} \leq C \frac{h_1^{|\alpha|} e^{-M(k|x|)} e^{M(2m|x|)} M_\alpha}{h^{|\alpha|} M_\alpha} \leq C,$$

hence  $\psi * F$  is a multiplier for  $\mathcal{S}'^{\{M_p\}}$  and the proof is complete.  $\square$

For  $S \in B^*$ , by lemma 4.1, for  $k > 0$ ,  $\frac{\cosh(k|x|)}{\cosh(2k|x|)} \in \mathcal{S}^*$  and by lemma 4.2 we have

$$\cosh(k|x|)S = \frac{\cosh(k|x|)}{\cosh(2k|x|)} \cosh(2k|x|)S \in \mathcal{O}_C'^*.$$

Similarly as in the proof of lemma 4.1 one can prove that  $(\cosh(k|x|))^{-1} \in \mathcal{S}^*$ , for  $k > 0$ . So, for  $S \in B^*$ , we also have  $S = (\cosh(k|x|))^{-1} \cosh(k|x|)S \in \mathcal{O}_C'^*$ . Using this, we get

$$B^* = \{S \in \mathcal{D}'^* | \cosh(k|x|)S \in \mathcal{O}_C'^*, \forall k \geq 0\}. \quad (16)$$

**Lemma 4.3.**  $\mathcal{O}_C'^{(M_p)} \subseteq \mathcal{D}_{L^1}'^{(M_p)}$  and  $\mathcal{O}_C'^{\{M_p\}} \subseteq \tilde{\mathcal{D}}_{L^1}'^{\{M_p\}}$ .

*Proof.* We will give the proof only in the  $\{M_p\}$  case, the  $(M_p)$  case is similar. Let  $S \in \mathcal{O}_C^{\{M_p\}}$ . From proposition 2 of [4], there exist  $k > 0$  and  $\{M_p\}$ -ultradifferential operator  $P(D)$  such that  $S = P(D)F_1 + F_2$  where  $\|e^{M(k|\cdot|)}(|F_1(x)| + |F_2(x)|)\|_{L^\infty} < \infty$ . We will assume that  $F_2 = 0$  and put  $F = F_1$ . The general case is proved analogously. Let  $\varphi \in \mathcal{D}^{\{M_p\}}$ . We have

$$|\langle S, \varphi \rangle| = |\langle F, P(-D)\psi \rangle| \leq \|e^{M(k|\cdot|)}F\|_{L^\infty} \|e^{-M(k|\cdot|)}\|_{L^1} \|P(-D)\varphi\|_{L^\infty} \leq Cp_{(t_j)}(\varphi),$$

for some  $C > 0$  and  $(t_j) \in \mathfrak{R}$ , where, the last inequality follows from the fact that  $P(D) : \dot{\mathcal{B}}^{\{M_p\}} \rightarrow \dot{\mathcal{B}}^{\{M_p\}}$  is continuous. Because  $\mathcal{D}^{\{M_p\}}$  is dense in  $\dot{\mathcal{B}}^{\{M_p\}}$ , the claim in the lemma follows.  $\square$

If we use the previous lemma in (16), we get

$$B^{(M_p)} = \left\{ S \in \mathcal{D}'^{(M_p)} \mid \cosh(k|x|)S \in \mathcal{D}'_{L^1}^{(M_p)}, \forall k \geq 0 \right\}, \quad (17)$$

$$B^{\{M_p\}} = \left\{ S \in \mathcal{D}'^{\{M_p\}} \mid \cosh(k|x|)S \in \tilde{\mathcal{D}}'_{L^1}^{\{M_p\}}, \forall k \geq 0 \right\}. \quad (18)$$

Now we will give the theorem that characterizes the elements of  $\mathcal{D}'^*$  for which the convolution with  $e^{s|x|^2}$  exists as an element of  $\mathcal{D}'^*$ .

**Theorem 4.3.** *Let  $s \in \mathbb{R}$ ,  $s \neq 0$ . Then*

- a) *The convolution of  $S \in \mathcal{D}'^*$  and  $e^{s|x|^2}$  exists if and only if  $S \in B_s^*$ .*
- b)  *$\mathcal{L} : B^* \rightarrow A^*$  is well defined and bijective mapping. For  $S \in B^*$  and  $\xi, \eta \in \mathbb{R}^d$ ,  $e^{-(\xi+i\eta)x}S(x) \in \mathcal{D}'_{L^1}^{(M_p)}(\mathbb{R}_x^d)$ , resp.  $e^{-(\xi+i\eta)x}S(x) \in \tilde{\mathcal{D}}'_{L^1}^{\{M_p\}}(\mathbb{R}_x^d)$  and the Laplace transform of  $S$  is given by  $\mathcal{L}(S)(\xi + i\eta) = \langle e^{-(\xi+i\eta)x}S(x), 1_x \rangle$ .*
- c) *The mapping  $B_s^* \rightarrow A_s^*$ ,  $S \mapsto S * e^{s|x|^2}$  is bijective and for  $S \in B_s^*$ ,  $(S * e^{s|x|^2})(x) = e^{s|x|^2} \mathcal{L}(e^{s|\cdot|^2}S)(2sx)$ .*

*Proof.* First we will prove a). Let  $S \in B_s^*$ . Let  $\varphi \in \mathcal{D}^*$  is fixed and  $K \subset \subset \mathbb{R}^d$ , such that  $\text{supp } \varphi \subseteq K$ . Note that

$$(\varphi * e^{s|\cdot|^2})(x) = e^{s|x|^2} \int_{\mathbb{R}^d} \varphi(y) e^{s|y|^2 - 2sxy} dy$$

and define  $f(x) = (\cosh(k|x|))^{-1} \int_{\mathbb{R}^d} \varphi(y) e^{s|y|^2 - 2sxy} dy$  where  $k$  will be chosen later. Put  $l = \sup\{|y| \mid y \in K\}$  to simplify notations. We will prove that  $f \in \mathcal{D}_{L^\infty}^*$ , for large enough  $k$ . For  $w \in \mathbb{C}^d$ , put  $g(w) = \int_{\mathbb{R}^d} \varphi(y) e^{s|y|^2 - 2swy} dy$ . Then  $g(w)$  is an entire function. To estimate its derivatives we use the Cauchy integral formula and obtain

$$|\partial^\alpha g(x)| \leq \frac{\alpha!}{r^{|\alpha|}} \sup_{|w_1 - x_1| \leq r, \dots, |w_d - x_d| \leq r} |g(w)|.$$

Take  $r < 1/(2dl|s|)$ . We put  $w = \xi + i\eta$  and estimate

$$\begin{aligned} |g(w)| &\leq \int_{\mathbb{R}^d} |\varphi(y)| e^{s|y|^2 - 2s\xi y} dy \leq e^{2|s||\xi|l} \|\varphi\|_{L^\infty} \int_K e^{s|y|^2} dy = c'' \|\varphi\|_{L^\infty} e^{2l|s||\xi|} \\ &\leq c'' \|\varphi\|_{L^\infty} e^{2l|s|(|x|+|\xi-x|)} = c'' \|\varphi\|_{L^\infty} e^{2l|s||\xi-x|} e^{2l|s||x|} \leq 3c'' \|\varphi\|_{L^\infty} e^{2l|s||x|}, \end{aligned}$$

where we denote  $c'' = \int_K e^{s|y|^2} dy$ . Hence, we get

$$|\partial_x^\alpha g(x)| \leq \frac{3c'' \|\varphi\|_{L^\infty} \alpha!}{r^{|\alpha|}} e^{2l|s||x|}. \quad (19)$$

We can use the same methods as in the proof of lemma 4.1 to prove that

$$\left| D^\alpha \left( \frac{1}{\cosh(k|x|)} \right) \right| \leq C \frac{\alpha!}{r^{|\alpha|}} e^{-c'k|x|}$$

for some  $C > 0$ ,  $c' > 0$  and  $c'$  doesn't depend on  $k$ . If we take  $r > 0$  small enough we can make it the same for (19) and the above estimate. Now take  $k$  large enough such that  $2l|s| < c'k$ . Then, for  $h > 0$  fixed, we have

$$\begin{aligned} \frac{h^{|\alpha|} |D^\alpha f(x)|}{M_\alpha} &\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \frac{h^{|\alpha|} |D^{\alpha-\beta} g(x)| |D^\beta ((\cosh(k|x|))^{-1})|}{M_\alpha} \\ &\leq 3c'' C \|\varphi\|_{L^\infty} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \frac{(2h)^{|\alpha|} (\alpha - \beta)! e^{2l|s||x|} \beta! e^{-c'k|x|}}{2^{|\alpha|} r^{|\alpha-\beta|} r^{|\beta|} M_\alpha} \\ &\leq \frac{3c'' C \|\varphi\|_{L^\infty}}{2^{|\alpha|}} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \left( \frac{2h}{r} \right)^{|\alpha|} \frac{\alpha!}{M_\alpha} e^{(2l|s| - c'k)|x|} \leq c'' C' \|\varphi\|_{L^\infty}, \end{aligned}$$

where we use the fact  $\frac{k^p p!}{M_p} \rightarrow 0$  when  $p \rightarrow \infty$ . From the arbitrariness of  $h$  we have  $f \in \mathcal{D}_{L^\infty}^*$ . Because  $\mathcal{D}_{L^\infty}^{\{M_p\}} = \tilde{\mathcal{D}}_{L^\infty}^{\{M_p\}}$  as a set,  $f \in \tilde{\mathcal{D}}_{L^\infty}^{\{M_p\}}$ . Now, we obtain

$$\left( \varphi * e^{s|\cdot|^2} \right) (x) S = f(x) \cosh(k|x|) e^{s|x|^2} S.$$

$e^{s|x|^2} S \in B^*$  (because  $S \in B_s^*$ ), hence, by (17), resp. (18),  $\cosh(k|x|) e^{s|x|^2} S \in \mathcal{D}_{L^1}^{\prime(M_p)}$ , resp.  $\cosh(k|x|) e^{s|x|^2} S \in \tilde{\mathcal{D}}_{L^1}^{\{M_p\}}$ . Hence  $(\varphi * e^{s|\cdot|^2}) S \in \mathcal{D}_{L^1}^{\prime(M_p)}$ , resp.  $(\varphi * e^{s|\cdot|^2}) S \in \tilde{\mathcal{D}}_{L^1}^{\{M_p\}}$ . The theorem of [5] implies that the convolution of  $S$  and  $e^{s|\cdot|^2}$  exists, in the  $(M_p)$  case. Let us consider the  $\{M_p\}$  case. If we prove that for arbitrary compact subset  $K$  of  $\mathbb{R}^d$ , the bilinear mapping  $(\varphi, \chi) \mapsto \left\langle \left( \varphi * e^{s|\cdot|^2} \right) S, \chi \right\rangle$ ,  $\mathcal{D}_K^{\{M_p\}} \times \dot{\mathcal{B}}^{\{M_p\}} \rightarrow \mathbb{C}$ , is continuous, theorem 1 of [14] will imply the existence of convolution of  $S$  and  $e^{s|\cdot|^2}$ . Let  $K \subset \subset \mathbb{R}^d$  be fixed. By the above consideration, we have

$$\left| \left\langle \left( \varphi * e^{s|\cdot|^2} \right) S, \chi \right\rangle \right| = \left| \left\langle \cosh(k|x|) e^{s|x|^2} S(x), f(x) \chi(x) \right\rangle \right| \leq C_1 p_{(t_j)}(f\chi),$$

for some  $C_1 > 0$  and  $(t_j) \in \mathfrak{R}$ , where, in the last inequality, we used that  $\cosh(k|x|)e^{s|x|^2}S \in \tilde{\mathcal{D}}_{L^1}^{\{M_p\}}$ . For brevity, denote  $T_\alpha = \prod_{j=1}^{|\alpha|} t_j$  and  $T_0 = 1$ . Observe that

$$\begin{aligned} \frac{|D^\alpha(f(x)\chi(x))|}{T_\alpha M_\alpha} &\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \frac{|D^\beta f(x)| |D^{\alpha-\beta} \chi(x)|}{T_\beta M_\beta T_{\alpha-\beta} M_{\alpha-\beta}} \leq \tilde{C} c'' \|\varphi\|_{L^\infty} p_{(t_j/2)}(\chi) \\ &\leq \tilde{C} c'' p_{(t_j/2), K}(\varphi) p_{(t_j/2)}(\chi), \end{aligned}$$

where we used the above estimates for the derivatives of  $f$ . Note that  $c''$  does not depend on  $\varphi$ , only on  $K$ . From this, the continuity of the bilinear mapping in consideration follows.

For the other direction, let the convolution of  $S$  and  $e^{s|x|^2}$  exists. Then, by the theorem of [5], resp. theorem 1 of [14], for every  $\varphi \in \mathcal{D}^*$ ,  $(\varphi * e^{s|\cdot|^2})S \in \mathcal{D}_{L^1}^{\{M_p\}}$ , resp.  $(\varphi * e^{s|\cdot|^2})S \in \tilde{\mathcal{D}}_{L^1}^{\{M_p\}}$ . Let  $\varphi \in \mathcal{D}^*$ , such that  $\varphi(y) \geq 0$ . Put  $U = \{y \in \mathbb{R}^d | \varphi(y) \neq 0\}$  and  $t = \sup\{|y| | y \in \text{supp } \varphi\}$ . Then we have

$$\int_{\mathbb{R}^d} \varphi(y) e^{s|y|^2 - 2sxy} dy \geq c e^{\inf_{y \in U} (-2sxy)},$$

where  $c = \int_{\mathbb{R}^d} \varphi(y) e^{s|y|^2} dy$ . Let  $x_0 \in \mathbb{R}^d$  and  $\varepsilon > 0$  be fixed. There exists  $\varphi \in \mathcal{D}^*$ , such that  $U \subseteq B(x_0, \varepsilon)$  ( $B(x_0, \varepsilon)$  is the ball in  $\mathbb{R}^d$  with center at  $x_0$  and radius  $\varepsilon$ ). Then

$$\inf_{y \in U} (-2sxy) \geq \inf_{y \in B(x_0, \varepsilon)} (-2sxy) = -2sxx_0 + \inf_{y \in B(x_0, \varepsilon)} (-2sx(y - x_0)) \geq -2sxx_0 - 2\varepsilon |s||x|.$$

We get

$$\int_{\mathbb{R}^d} \varphi(y) e^{s|y|^2 - 2sxy} dy \geq c e^{-2sxx_0 - 2\varepsilon |s||x|}.$$

Define  $f(x) = e^{-2sxx_0 - 2\varepsilon |s|\sqrt{1+|x|^2}} \left( \int_{\mathbb{R}^d} \varphi(y) e^{s|y|^2 - 2sxy} dy \right)^{-1}$ . We will prove that  $f \in \mathcal{D}_{L^\infty}^*$ .

$g(w) = \int_{\mathbb{R}^d} \varphi(y) e^{s|y|^2 - 2swy} dy$  is an entire function. Put  $w = \xi + i\eta$ . Then, for  $w$  in the strip  $\mathbb{R}^d + i\{\eta \in \mathbb{R}^d | |\eta| < 1/(8|s|t)\}$  and  $y \in \text{supp } \varphi$ , we have  $|2s\eta y| \leq 2|s||\eta||y| \leq 1/4 < \pi/4$ , hence

$$\left| \int_{\mathbb{R}^d} \varphi(y) e^{s|y|^2 - 2swy} dy \right| \geq \left| \int_{\mathbb{R}^d} \varphi(y) e^{s|y|^2 - 2s\xi y} \cos(2s\eta y) dy \right| \geq \frac{1}{\sqrt{2}} \int_{\mathbb{R}^d} \varphi(y) e^{s|y|^2 - 2s\xi y} dy > 0.$$

Moreover,  $e^{-2swx_0 - 2\varepsilon |s|\sqrt{1+w^2}}$  is analytic on the strip  $\mathbb{R}_\xi^d + i\{\eta \in \mathbb{R}^d | |\eta| < 1/4\}$ , where we take the principal branch of the square root which is single valued and analytic on  $\mathbb{C} \setminus (-\infty, 0]$ . So, for  $r_0 = \min\{1/4, 1/(8|s|t)\}$ ,  $f(w)$  is analytic on the strip  $\mathbb{R}^d + i\{\eta \in \mathbb{R}^d | |\eta| < r_0\}$ . To estimate the derivatives of  $f$ , we use Cauchy integral formula and obtain

$$|\partial^\alpha f(x)| \leq \frac{\alpha!}{r^{|\alpha|}} \sup_{|w_1 - x_1| \leq r, \dots, |w_d - x_d| \leq r} |f(w)|, \quad (20)$$



where  $r < r_0/(2d)$ . Put  $\rho = \sqrt{(1 + |\xi|^2 - |\eta|^2)^2 + 4(\xi\eta)^2}$ ,  $\cos \theta = \frac{1 + |\xi|^2 - |\eta|^2}{\sqrt{(1 + |\xi|^2 - |\eta|^2)^2 + 4(\xi\eta)^2}}$

and  $\sin \theta = \frac{2\xi\eta}{\sqrt{(1 + |\xi|^2 - |\eta|^2)^2 + 4(\xi\eta)^2}}$ , where  $\theta \in (-\pi, \pi)$ , from what it follows that  $\theta \in (-\pi/2, \pi/2)$  (because  $\cos \theta > 0$  and  $\theta \in (-\pi, \pi)$ ). Then

$$\begin{aligned} \operatorname{Re} \left( \sqrt{1+w^2} \right) &= \operatorname{Re} \left( \sqrt{\rho} \left( \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right) \right) = \sqrt{\rho} \cos \frac{\theta}{2} = \sqrt{\rho} \sqrt{\frac{\cos \theta + 1}{2}} \\ &= \frac{\sqrt{\rho \cos \theta + \rho}}{\sqrt{2}} = \frac{1}{\sqrt{2}} \sqrt{1 + |\xi|^2 - |\eta|^2 + \sqrt{(1 + |\xi|^2 - |\eta|^2)^2 + 4(\xi\eta)^2}} \\ &\geq \frac{1}{\sqrt{2}} \sqrt{1 + |\xi|^2 - |\eta|^2 + 1 + |\xi|^2 - |\eta|^2} = \sqrt{1 + |\xi|^2 - |\eta|^2}, \end{aligned}$$

where the first equality follows from the fact that we take the principal branch of the square root. We obtain

$$\begin{aligned} |f(w)| &= \frac{\left| e^{-2s w x_0 - 2\varepsilon |s| \sqrt{1+w^2}} \right|}{\left| \int_{\mathbb{R}^d} \varphi(y) e^{s|y|^2 - 2s w y} dy \right|} \leq \frac{\sqrt{2} e^{-2s \xi x_0} e^{-2\varepsilon |s| \operatorname{Re}(\sqrt{1+w^2})}}{\int_{\mathbb{R}^d} \varphi(y) e^{s|y|^2 - 2s \xi y} dy} \\ &\leq \frac{\sqrt{2} e^{-2s \xi x_0} e^{-2\varepsilon |s| \sqrt{1+|\xi|^2-|\eta|^2}}}{\int_{\mathbb{R}^d} \varphi(y) e^{s|y|^2 - 2s \xi y} dy} \leq \frac{\sqrt{2} e^{-2s \xi x_0} e^{-2\varepsilon |s| |\xi|}}{c e^{-2s \xi x_0 - 2\varepsilon |s| |\xi|}} \leq C_0'''. \end{aligned}$$

So, from (20), we have  $|\partial_x^\alpha f(x)| \leq C_0 \alpha! / r^{|\alpha|}$ , for some  $C_0 > 0$ . From this it easily follows that  $f \in \mathcal{D}_{L^\infty}^*$ . Now we have

$$e^{-2s x x_0 - 2\varepsilon |s| \sqrt{1+|x|^2}} e^{s|x|^2} S = f(x) \left( \varphi * e^{s|\cdot|^2} \right) (x) S \in \mathcal{D}_{L^1}^{\prime(M_p)}, \text{ resp.} \quad (21)$$

$$e^{-2s x x_0 - 2\varepsilon |s| \sqrt{1+|x|^2}} e^{s|x|^2} S = f(x) \left( \varphi * e^{s|\cdot|^2} \right) (x) S \in \tilde{\mathcal{D}}_{L^1}^{\{M_p\}}, \quad (22)$$

where we used the fact that  $(\varphi * e^{s|\cdot|^2}) S \in \mathcal{D}_{L^1}^{\prime(M_p)}$ , resp.  $(\varphi * e^{s|\cdot|^2}) S \in \tilde{\mathcal{D}}_{L^1}^{\{M_p\}}$  (which, as noted before, follows from the existence of the convolution of  $S$  and  $e^{s|\cdot|^2}$ ) and these hold for every  $x_0 \in \mathbb{R}^d$  and every  $\varepsilon > 0$ . Now, put  $x'_0 = 2s x_0$ ,  $x''_0 = -2s x_0$  and  $\varepsilon' = 2|s|\varepsilon$ . Then, from (21), resp. (22), we have

$$\begin{aligned} e^{-x x'_0 - \varepsilon' \sqrt{1+|x|^2}} e^{s|x|^2} S &\in \mathcal{D}_{L^1}^{\prime(M_p)} \quad , \quad e^{x x''_0 - \varepsilon' \sqrt{1+|x|^2}} e^{s|x|^2} S \in \mathcal{D}_{L^1}^{\prime(M_p)}, \text{ resp.} \\ e^{-x x'_0 - \varepsilon' \sqrt{1+|x|^2}} e^{s|x|^2} S &\in \tilde{\mathcal{D}}_{L^1}^{\{M_p\}} \quad , \quad e^{x x''_0 - \varepsilon' \sqrt{1+|x|^2}} e^{s|x|^2} S \in \tilde{\mathcal{D}}_{L^1}^{\{M_p\}} \end{aligned}$$

and from arbitrariness of  $x_0$  and  $\varepsilon > 0$  it follows

$$\cosh(x x_0) e^{-\varepsilon \sqrt{1+|x|^2}} e^{s|x|^2} S \in \mathcal{D}_{L^1}^{\prime(M_p)}, \text{ resp. } \cosh(x x_0) e^{-\varepsilon \sqrt{1+|x|^2}} e^{s|x|^2} S \in \tilde{\mathcal{D}}_{L^1}^{\{M_p\}} \quad (23)$$

for all  $x_0 \in \mathbb{R}^d$  and all  $\varepsilon > 0$ . Let  $l > 0$ . Take  $x^{(j)} \in \mathbb{R}^d$ ,  $j = 1, \dots, d$ , to be such that  $x_q^{(j)} = 0$ , for  $j \neq q$  and  $x_j^{(j)} = ld$ . Then

$$\cosh(l|x|) \leq e^{l|x|} \leq \prod_{j=1}^d e^{l|x_j|} \leq \left( \sum_{j=1}^d \frac{1}{d} e^{l|x_j|} \right)^d \leq \sum_{j=1}^d e^{ld|x_j|} \leq 2 \sum_{j=1}^d \cosh(x^{(j)}x). \quad (24)$$

We will prove that  $\cosh(l|x|) \left( \sum_{j=1}^d \cosh(2x^{(j)}x) \right)^{-1} \in \mathcal{D}_{L^\infty}^*$ . The function  $\sum_{j=1}^d \cosh(2ldw_j)$  is an entire function of  $w = \xi + i\eta$ . Moreover, for  $w \in U = \mathbb{R}_\xi^d + i\{\eta \in \mathbb{R}^d \mid |\eta| < 1/(4ld^2)\}$ , we have

$$\begin{aligned} \left| \sum_{j=1}^d \cosh(2ldw_j) \right| &= \frac{1}{2} \left| \sum_{j=1}^d (e^{2ld\xi_j} + e^{-2ld\xi_j}) \cos(2ld\eta_j) + i \sum_{j=1}^d (e^{2ld\xi_j} - e^{-2ld\xi_j}) \sin(2ld\eta_j) \right| \\ &\geq \frac{1}{2} \left| \sum_{j=1}^d (e^{2ld\xi_j} + e^{-2ld\xi_j}) \cos(2ld\eta_j) \right| \geq \frac{\sqrt{2}}{4} \sum_{j=1}^d (e^{2ld\xi_j} + e^{-2ld\xi_j}) \\ &\geq \frac{\sqrt{2}}{4} \sum_{j=1}^d e^{2ld|\xi_j|}, \end{aligned}$$

hence

$$\left| \sum_{j=1}^d \cosh(2ldw_j) \right| \geq \frac{\sqrt{2}}{4} \sum_{j=1}^d e^{2ld|\xi_j|} > 0, \text{ for all } w = \xi + i\eta \in U. \quad (25)$$

For  $\cosh(l|x|)$ , we already proved that is the restriction to  $\mathbb{R}^d \setminus \{0\}$  of the function  $\cosh(l\sqrt{w^2})$  which is analytic on  $W = \{w = \xi + i\eta \in \mathbb{C}^d \mid |\xi| > 2|\eta|\}$  (see the proof of lemma 4.1). Hence

$\cosh(l\sqrt{w^2}) \left( \sum_{j=1}^d \cosh(2ldw_j) \right)^{-1}$  is analytic on  $W \cap U$ . We will use the same notations

that were used in the proof of lemma 4.1. Similarly as there, put  $g_k(w) = \sum_{n=0}^{\infty} \frac{k^{2n}(w^2)^n}{(2n)!}$ .

Then  $g_k(w) = (e^{k\sqrt{w^2}} + e^{-k\sqrt{w^2}})/2$ , for  $w \in W_r \cap \mathbb{R}_\xi^d$  and from the uniqueness of analytic continuation and arbitrariness of  $r > 0$  it follows  $g_k(w) = (e^{k\sqrt{w^2}} + e^{-k\sqrt{w^2}})/2$  on  $W$ . Fix

$0 < r_0 < 1/(8ld^3)$ . Then, for  $w \in \overline{B(0, r_0)}$ , by (25), we have  $\left| g_l(w) \left( \sum_{j=1}^d \cosh(2ldw_j) \right)^{-1} \right| \leq$

$C_{r_0}$ . Take  $r_1 > 0$  such that  $\overline{B(x, 2dr_1)} \subseteq (\mathbb{C}^d \setminus \overline{B(0, r_0/16)}) \cap W \cap U$ , for all  $x \in W_{\frac{r_0}{4}} \cap \mathbb{R}_x^d$ . For such  $x$ , we use Cauchy integral formula to estimate

$$\left| \partial^\alpha \left( \frac{\cosh(l\sqrt{x^2})}{\sum_{j=1}^d \cosh(2ldx_j)} \right) \right| \leq \frac{\alpha!}{r_1^\alpha} \sup_{|w_1 - x_1| \leq r_1, \dots, |w_d - x_d| \leq r_1} \left| \frac{\cosh(l\sqrt{w^2})}{\sum_{j=1}^d \cosh(2ldw_j)} \right|.$$

Now, using (25), we have

$$\begin{aligned} \left| \frac{\cosh(l\sqrt{w^2})}{\sum_{j=1}^d \cosh(2ldw_j)} \right| &\leq \frac{2}{\sqrt{2}} \frac{e^{l\operatorname{Re}\sqrt{w^2}} + e^{-l\operatorname{Re}\sqrt{w^2}}}{\sum_{j=1}^d e^{2ld|\xi_j|}} \leq \frac{4e^{l\sqrt{(|\xi|^2-|\eta|^2)+4(\xi\eta)^2}}}{\sum_{j=1}^d e^{2ld|\xi_j|}} \\ &\leq \frac{4e^{l\sqrt{|\xi|^2-|\eta|^2+2|\xi\eta|}}}{\sum_{j=1}^d e^{2ld|\xi_j|}} \leq \frac{4e^{2l|\xi|}}{\sum_{j=1}^d e^{2ld|\xi_j|}} \leq \frac{8\cosh(2l|\xi|)}{\sum_{j=1}^d e^{2ld|\xi_j|}} \leq C', \end{aligned}$$

where the last inequality follows from (24). Hence, for  $x \in W_{\frac{r_0}{4}} \cap \mathbb{R}_x^d$  we get

$$\left| \partial^\alpha \left( \frac{\cosh(l|x|)}{\sum_{j=1}^d \cosh(2ldx_j)} \right) \right| \leq C' \frac{\alpha!}{r_1^\alpha}.$$

For  $x \in (B(0, r_0/2) \cap \mathbb{R}_x^d) \setminus \{0\}$ , if we take  $r_2 > 0$  small enough such that  $\overline{B(x, 2dr_2)} \subseteq B(0, r_0)$  we have (from Cauchy integral formula)

$$\begin{aligned} \left| \partial^\alpha \left( \frac{\cosh(l\sqrt{x^2})}{\sum_{j=1}^d \cosh(2ldx_j)} \right) \right| &= \left| \partial^\alpha \left( \frac{g_l(x)}{\sum_{j=1}^d \cosh(2ldx_j)} \right) \right| \\ &\leq \frac{\alpha!}{r_2^\alpha} \sup_{|w_1-x_1| \leq r_2, \dots, |w_d-x_d| \leq r_2} \left| \frac{g_l(w)}{\sum_{j=1}^d \cosh(2ldw_j)} \right| \leq C_{r_0} \frac{\alpha!}{r_2^\alpha}. \end{aligned}$$

Because  $\cosh(l|x|) \left( \sum_{j=1}^d \cosh(2x^{(j)}x) \right)^{-1}$  is in  $\mathcal{C}^\infty(\mathbb{R}^d)$  the same inequality will hold for the derivatives in  $x = 0$ . If we take  $r = \min\{r_1, r_2\}$  we get that, for  $x \in \mathbb{R}^d$ ,

$$\left| \partial_x^\alpha \left( \frac{\cosh(l|x|)}{\sum_{j=1}^d \cosh(2x^{(j)}x)} \right) \right| \leq C \frac{\alpha!}{r^\alpha}.$$

Now, it easily follows that  $\cosh(l|x|) \left( \sum_{j=1}^d \cosh(2x^{(j)}x) \right)^{-1} \in \mathcal{D}_{L^\infty}^*$ . From (23), we have

$$\cosh(l|x|)e^{-\varepsilon\sqrt{1+|x|^2}}e^{s|x|^2}S \in \mathcal{D}_{L^1}^{\prime(M_p)}, \text{ resp. } \cosh(l|x|)e^{-\varepsilon\sqrt{1+|x|^2}}e^{s|x|^2}S \in \tilde{\mathcal{D}}_{L^1}^{\prime\{M_p\}}, \quad (26)$$

for every  $l > 0$  and every  $\varepsilon > 0$ . Let  $l > 0$  be fixed. By considering the function  $e^{\varepsilon\sqrt{1+z^2}}$ , which is analytic on the strip  $\mathbb{R}^d + i\{y \in \mathbb{R}^d | |y| < 1/4\}$ , we obtain the estimates  $\left| \partial^\alpha e^{\varepsilon\sqrt{1+|x|^2}} \right| \leq \tilde{C} \frac{\alpha!}{\tilde{r}^{|\alpha|}} e^{2\varepsilon\sqrt{1+|x|^2}}$ , for  $\tilde{r} < 1/(8d)$  and some  $\tilde{C} > 0$ . By this and (15), for small enough  $r > 0$ , we have

$$\left| D^\alpha \left( \frac{\cosh\left(\frac{l|x|}{2}\right)}{\cosh(l|x|)} e^{\varepsilon\sqrt{1+|x|^2}} \right) \right| \leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \left| D^\beta \left( \frac{\cosh\left(\frac{l|x|}{2}\right)}{\cosh(l|x|)} \right) \right| \left| D^{\alpha-\beta} e^{\varepsilon\sqrt{1+|x|^2}} \right|$$

$$\begin{aligned}
&\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} C' \frac{\beta!}{r^{|\beta|}} e^{-\frac{\varepsilon}{2}l|x|} \frac{(\alpha - \beta)!}{r^{|\alpha| - |\beta|}} e^{2\varepsilon\sqrt{1+|x|^2}} \\
&\leq C' \frac{\alpha!}{r^{|\alpha|}} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} e^{-\frac{\varepsilon}{2}l|x|} e^{2\varepsilon\sqrt{1+|x|^2}} \leq C'' \alpha! \left(\frac{2}{r}\right)^{|\alpha|},
\end{aligned}$$

where the last inequality will hold if we take  $\varepsilon < cl/4$  and  $c$  is the one defined in the proof of lemma 4.1. We get that  $\frac{\cosh(\frac{l}{2}|x|)}{\cosh(l|x|)} e^{\varepsilon\sqrt{1+|x|^2}} \in \mathcal{D}_{L^\infty}^*$ . From this and (26) we get  $\cosh(\frac{l}{2}|x|) e^{s|x|^2} S \in \mathcal{D}_{L^1}^{\prime(M_p)}$ , resp.  $\cosh(\frac{l}{2}|x|) e^{s|x|^2} S \in \tilde{\mathcal{D}}_{L^1}^{\prime\{M_p\}}$ . From the arbitrariness of  $l > 0$ , we have

$$\cosh(l|x|) e^{s|x|^2} S \in \mathcal{D}_{L^1}^{\prime(M_p)}, \text{ resp. } \cosh(l|x|) e^{s|x|^2} S \in \tilde{\mathcal{D}}_{L^1}^{\prime\{M_p\}}$$

for all  $l > 0$ . By (17), resp. (18), we have that  $e^{s|x|^2} S \in B^*$ . Hence  $S \in B_s^*$ .

Let us prove b). Let  $S \in B^*$ . Similarly as in the proof of lemma 4.1, we can prove that for each fixed  $\xi \in \mathbb{R}^d$  there exists  $k_\xi > 0$  ( $k$  depends on  $\xi$ ) such that  $\frac{e^{-x\xi}}{\cosh(k_\xi|x|)} \in \mathcal{S}^*(\mathbb{R}_x^d)$ . Then, for fixed  $\xi \in \mathbb{R}^d$ , we have

$$e^{-x\xi} S = \frac{e^{-x\xi}}{\cosh(k_\xi|x|)} \cosh(k_\xi|x|) S \in \mathcal{S}'^*(\mathbb{R}_x^d).$$

Hence, by theorem 4.1, the Laplace transform of  $S$  exists and belongs to  $A^*$ . Analogously, for  $\varepsilon > 0$  and  $\xi + i\eta$  fixed, we can find  $k > 0$  ( $k$  depends on  $\varepsilon$  and  $\xi + i\eta$ ) such that  $\frac{e^{-(\xi+i\eta)x} e^{\varepsilon\sqrt{1+|x|^2}}}{\cosh(k|x|)} \in \mathcal{S}^*(\mathbb{R}_x^d)$ . Then

$$e^{-(\xi+i\eta)x} e^{\varepsilon\sqrt{1+|x|^2}} S = \frac{e^{-(\xi+i\eta)x} e^{\varepsilon\sqrt{1+|x|^2}}}{\cosh(k|x|)} \cosh(k|x|) S \in \mathcal{D}_{L^1}^{\prime(M_p)}(\mathbb{R}_x^d),$$

in the  $(M_p)$  case and resp.  $e^{-(\xi+i\eta)x} e^{\varepsilon\sqrt{1+|x|^2}} S \in \tilde{\mathcal{D}}_{L^1}^{\prime\{M_p\}}$  in the  $\{M_p\}$  case. By (11), we have

$$\mathcal{L}(S)(\xi + i\eta) = \left\langle e^{\varepsilon\sqrt{1+|x|^2}} e^{-(\xi+i\eta)x} S(x), e^{-\varepsilon\sqrt{1+|x|^2}} \right\rangle = \langle e^{-(\xi+i\eta)x} S(x), 1_x \rangle.$$

The injectivity is obvious. Let us prove the surjectivity. By theorem 4.2, for  $f \in A^*$  there exists  $T \in \mathcal{D}^*$  such that  $e^{-x\xi} T(x) \in \mathcal{S}'^*(\mathbb{R}_x^d)$ , for all  $\xi \in \mathbb{R}_\xi^d$  and  $\mathcal{L}(T)(\xi + i\eta) = f(\xi + i\eta)$ . Because  $e^{-x\xi} T(x) \in \mathcal{S}'^*(\mathbb{R}_x^d)$ , for all  $\xi \in \mathbb{R}^d$  we obtain that  $\cosh(x\xi) T(x) \in \mathcal{S}'^*(\mathbb{R}_x^d)$  for all  $\xi \in \mathbb{R}^d$ . Let  $k > 0$ . By the considerations in the proof of a), if take  $x^{(j)} \in \mathbb{R}^d$ ,  $j = 1, \dots, d$ , such that  $x_q^{(j)} = 0$ , for  $j \neq q$  and  $x_j^{(j)} = kd$ , we obtain that

$$\cosh(k|x|) \left( \sum_{j=1}^d \cosh(2x^{(j)}x) \right)^{-1} \in \mathcal{D}_{L^\infty}^*. \text{ Obviously } \mathcal{D}_{L^\infty}^* \subseteq \mathcal{O}_M^*. \text{ Hence}$$

$$\cosh(k|x|) T(x) = \cosh(k|x|) \left( \sum_{j=1}^d \cosh(2x^{(j)}x) \right)^{-1} \sum_{j=1}^d \cosh(2x^{(j)}x) T(x) \in \mathcal{S}'^*(\mathbb{R}^d).$$

We obtain  $T \in B^*$  and the surjectivity is proved.

Now we will prove c). By a),  $S * e^{s|\cdot|^2}$  is well defined for  $S \in B_s^*$ . Let  $\psi \in \mathcal{D}^*$  is such that  $0 \leq \psi \leq 1$ ,  $\psi(x) = 1$  when  $|x| \leq 1$  and  $\psi(x) = 0$  when  $|x| > 2$ . Put  $\psi_j(x) = \psi(x/j)$  for  $j \in \mathbb{Z}_+$ . Because the convolution of  $S$  and  $e^{s|x|^2}$  exists,

$$\langle S * e^{s|\cdot|^2}, \varphi \rangle = \langle (\varphi * e^{s|\cdot|^2}) S, 1 \rangle = \lim_{j \rightarrow \infty} \langle (\varphi * e^{s|\cdot|^2}) S, \psi_j \rangle, \quad (27)$$

for all  $\varphi \in \mathcal{D}^*$ . Fix  $j \in \mathbb{Z}_+$  and observe that  $\langle (\varphi * e^{s|\cdot|^2}) S, \psi_j \rangle = \langle (\psi_j S) * e^{s|\cdot|^2}, \varphi \rangle$ . Let  $l \in \mathbb{N}$  be so large such that  $\text{supp } \psi_j \subseteq \{x \in \mathbb{R}^d | \psi_l(x) = 1\}$ . We have

$$\begin{aligned} & \langle (\varphi * e^{s|\cdot|^2}) S, \psi_j \rangle \\ &= \langle (\varphi * e^{s|\cdot|^2})(\xi) (\psi_j S)(\xi), \psi_l(\xi) \rangle = \left\langle e^{s|\xi|^2} \int_{\mathbb{R}^d} \varphi(x) e^{s|x|^2 - 2sx\xi} dx (\psi_j S)(\xi), \psi_l(\xi) \right\rangle \\ &= \left\langle e^{s|\xi|^2} e^{s|x|^2 - 2sx\xi} (\psi_j S)(\xi), \psi_l(\xi) \varphi(x) \right\rangle = \left\langle e^{s|x|^2} \left\langle e^{s|\xi|^2} e^{-2sx\xi} (\psi_j S)(\xi), \psi_l(\xi) \right\rangle, \varphi(x) \right\rangle \\ &= \left\langle e^{s|x|^2} \left\langle e^{s|\xi|^2} e^{-2sx\xi} S(\xi), \psi_j(\xi) \right\rangle, \varphi(x) \right\rangle, \end{aligned}$$

where the third and the fourth equality follow from theorem 2.3 of [7]. We obtain  $\langle (\psi_j S) * e^{s|\cdot|^2}, \varphi \rangle = \left\langle e^{s|x|^2} \left\langle e^{s|\xi|^2} e^{-2sx\xi} S(\xi), \psi_j(\xi) \right\rangle, \varphi(x) \right\rangle$ , for all  $\varphi \in \mathcal{D}^*$  and all  $j \in \mathbb{Z}_+$ . Hence

$$e^{s|x|^2} \left\langle e^{s|\xi|^2} e^{-2sx\xi} S(\xi), \psi_j(\xi) \right\rangle = \left( (\psi_j S) * e^{s|\cdot|^2} \right)(x) \quad (28)$$

in  $\mathcal{D}'^*(\mathbb{R}_x^d)$ , for all  $j \in \mathbb{Z}_+$ . Because  $\left\langle e^{s|\xi|^2} e^{-2sx\xi} S(\xi), \psi_j(\xi) \right\rangle = \left\langle \psi_j(\xi) S(\xi), e^{s|\xi|^2} e^{-2sx\xi} \right\rangle$ , for every fixed  $x \in \mathbb{R}^d$ , theorem 3.10 of [8] implies that the left hand side of (28) is an element of  $\mathcal{E}^*(\mathbb{R}_x^d)$ . By (27), the right hand side of (28) tends to  $S * e^{s|\cdot|^2}$  in  $\mathcal{D}'^*$ . Because  $S \in B_s^*$ ,  $e^{s|\cdot|^2} S \in B^*$  and by b), for each fixed  $x, y \in \mathbb{R}^d$ ,  $e^{-(x+iy) \cdot} e^{s|\cdot|^2} S \in \mathcal{D}'_{L^1}^{(M_p)}$ , resp.  $e^{-(x+iy) \cdot} e^{s|\cdot|^2} S \in \tilde{\mathcal{D}}_{L^1}^{\{M_p\}}$ , the Laplace transform of  $e^{s|\cdot|^2} S$  exists and  $\mathcal{L} \left( e^{s|\cdot|^2} S \right) (2sx) = \left\langle e^{s|\xi|^2} e^{-2sx\xi} S(\xi), 1_\xi \right\rangle$ , for every fixed  $x \in \mathbb{R}^d$ . So, the right hand side of (28) tends to  $e^{s|x|^2} \mathcal{L} \left( e^{s|\cdot|^2} S \right) (2sx)$  pointwise. We will prove that the convergence holds in  $\mathcal{D}'^*$ . Let  $K$  be a fixed compact subset of  $\mathbb{R}^d$ . With similar technic as in the proof of lemma 4.1, we can find large enough  $k > 0$  ( $k$  depends on  $K$ ) such that  $e^{-2sx\xi} (\cosh(k|\xi|))^{-1} \in \mathcal{S}^*(\mathbb{R}_\xi^d)$ , for each  $x \in K$  and the set  $\{e^{-2sx \cdot} (\cosh(k|\cdot|))^{-1} \in \mathcal{S}^*(\mathbb{R}_\xi^d) | x \in K\}$  is bounded subset of  $\mathcal{S}^*(\mathbb{R}_\xi^d)$ . Because  $S \in B_s^*$ ,  $\cosh(k|\cdot|) e^{s|\cdot|^2} S \in \mathcal{S}'^*$ . Hence

$$\begin{aligned} \left\langle e^{s|\xi|^2} e^{-2sx\xi} S(\xi), \psi_j(\xi) \right\rangle &= \left\langle e^{s|\xi|^2} e^{-2sx\xi} (\cosh(k|\xi|))^{-1} \cosh(k|\xi|) S(\xi), \psi_j(\xi) \right\rangle \\ &= \left\langle e^{s|\xi|^2} \cosh(k|\xi|) S(\xi), e^{-2sx\xi} (\cosh(k|\xi|))^{-1} \psi_j(\xi) \right\rangle. \end{aligned}$$

By the way we defined  $\psi_j$ , one easily verifies that  $\{e^{-2sx} \cdot (\cosh(k|\cdot|))^{-1} \psi_j(\cdot) | x \in K, j \in \mathbb{Z}_+\}$  is a bounded subset of  $\mathcal{S}^*(\mathbb{R}_\xi^d)$ . From this it follows that there exists  $C_K > 0$  ( $C_K$  depends on  $K$ ) such that  $\left| e^{s|x|^2} \left\langle e^{s|\xi|^2} e^{-2sx\xi} S(\xi), \psi_j(\xi) \right\rangle \right| \leq C_K$ , for all  $x \in K, j \in \mathbb{Z}_+$ . Because  $e^{s|x|^2} \left\langle e^{s|\xi|^2} e^{-2sx\xi} S(\xi), \psi_j(\xi) \right\rangle$  tends to  $e^{s|x|^2} \mathcal{L}(e^{s|\cdot|^2} S)(2sx)$  pointwise, by the above, the convergence also holds in  $\mathcal{D}'^*(\mathbb{R}_x^d)$ . Hence, we obtain  $e^{s|x|^2} \mathcal{L}(e^{s|\cdot|^2} S)(2sx) = (S * e^{s|\cdot|^2})(x)$ . Now,  $b$ ) implies  $S * e^{s|\cdot|^2} \in A_s^*$ . The bijectivity of  $S \mapsto S * e^{s|\cdot|^2}$  follows from the bijectivity of  $\mathcal{L} : B^* \rightarrow A^*$ .  $\square$

## 5 A new class of Anti-Wick operators

Theorem 4.3, along with (5), allows us to define Anti-Wick operators  $A_a : \mathcal{D}^*(\mathbb{R}^d) \rightarrow \mathcal{D}'^*(\mathbb{R}^d)$ , when  $a$  is not necessary in  $\mathcal{S}'^*(\mathbb{R}^{2d})$ . If  $a \in B_{-1}^*$  (and only then)  $b(x, \xi) = \pi^{-d} (a(\cdot, \cdot) * e^{-|\cdot|^2 - |\cdot|^2})(x, \xi)$  exists and is an element of  $A_{-1}^*$ . If this  $b$  is such that, for every  $\chi \in \mathcal{D}^*(\mathbb{R}^{2d})$  the integral

$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-y)\xi} b\left(\frac{x+y}{2}, \xi\right) \chi(x, y) dx dy d\xi \quad (29)$$

is well defined as oscillatory integral and  $\langle K_b, \chi \rangle$  defined as the above integral is well defined ultradistributions, then the operator associated to that kernel (see theorem 2.3 of [7])  $\varphi \mapsto \langle K_b(x, y), \varphi(y) \rangle$ ,  $\mathcal{D}^*(\mathbb{R}^d) \rightarrow \mathcal{D}'^*(\mathbb{R}^d)$ , can be called the Anti-Wick operator with symbol  $a$  (because of proposition 3.4, this is appropriate generalization of Anti-Wick operators). The next theorem gives an example of such  $b$ .

**Theorem 5.1.** *If  $a \in B_{-1}^*$  is such that  $b$ , given by (5), satisfies the following condition: for every  $K \subset\subset \mathbb{R}_x^d$  there exists  $\tilde{r} > 0$  such that there exist  $m, C_1 > 0$ , resp. there exist  $C_1 > 0$  and  $(k_p) \in \mathfrak{R}$ , (in both cases  $C_1$  and  $m$ , resp.  $C_1$  and  $(k_p)$  depend on  $K$ ) such that*

$$|b(x + i\eta, \xi)| \leq C_1 e^{M(m|\xi|)}, \text{ resp. } |b(x + i\eta, \xi)| \leq C_1 e^{N_{k_p}(|\xi|)}, x \in K, |\eta| < \tilde{r}, \xi \in \mathbb{R}^d, \quad (30)$$

*then (29) is oscillatory integral and  $K_b$  defined by (29) is well defined ultradistribution.*

*Proof.* Under the conditions in the theorem, Cauchy integral formula yields  $|D_x^\alpha b(x, \xi)| \leq C\alpha! / r_1^{|\alpha|} e^{M(m|\xi|)}$ , resp.  $|D_x^\alpha b(x, \xi)| \leq C\alpha! / r_1^{|\alpha|} e^{N_{k_p}(|\xi|)}$ , for all  $x \in K, \xi \in \mathbb{R}^d$  ( $r_1$  and  $C$  depend on  $K$ ). Let  $U$  be an arbitrary bounded open subset of  $\mathbb{R}^{2d}$ . Then  $V = \{t \in \mathbb{R}^d | t = (x+y)/2, (x, y) \in U\}$  is a bounded set in  $\mathbb{R}^d$ , hence  $K = \overline{V}$  is compact set. For this  $K$ , let  $m$ , resp.  $(k_p)$  be as in (30). Take  $P_l$ , resp.  $P_{l_p}$ , as in proposition 1.1, such that  $|P_l(\xi)| \geq C_2 e^{M(r|\xi|)}$ , resp.  $|P_{l_p}(\xi)| \geq C_2 e^{N_{r_p}(\xi)}$ , for some  $C_2 > 0$ , such that  $\int_{\mathbb{R}^d} e^{M(m|\xi|)} e^{-M(r|\xi|)} d\xi < \infty$ , resp.  $\int_{\mathbb{R}^d} e^{N_{k_p}(|\xi|)} e^{-N_{r_p}(|\xi|)} d\xi < \infty$ . We can define  $K_{b,U}$  as

$$\langle K_{b,U}, \chi \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{3d}} \frac{e^{i(x-y)\xi}}{P_l(\xi)} P_l(D_y) \left( b\left(\frac{x+y}{2}, \xi\right) \chi(x, y) \right) dx dy d\xi, \chi \in \mathcal{D}^*(U),$$

in the  $(M_p)$  case, resp. the same but with  $P_l$  in place of  $P_l$  in the  $\{M_p\}$  case and then one easily checks that  $K_{b,U} \in \mathcal{D}'^*(U)$ . Moreover, if  $\psi \in \mathcal{D}'^*(\mathbb{R}^d)$  is such that  $\psi(\xi) = 1$  in a neighborhood of 0, for  $\delta > 0$ , we can define  $K_{b,U,\psi,\delta} \in \mathcal{D}'^*(U)$  as

$$\langle K_{b,U,\psi,\delta}, \chi \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{3d}} e^{i(x-y)\xi} \psi(\delta\xi) b\left(\frac{x+y}{2}, \xi\right) \chi(x, y) dx dy d\xi.$$

Then  $K_{b,U,\psi,\delta} \rightarrow K_{b,U}$ , when  $\delta \rightarrow 0^+$ , in  $\mathcal{D}'^*(U)$ . Combining these results, we obtain that the definition of  $K_{b,U}$  does not depend on  $P_l$  resp.  $P_p$ , when these are appropriately chosen (see the above discussion) and on the choice of  $\psi$  with the above properties. Moreover, when  $U_1$  and  $U_2$  are two bounded open sets in  $\mathbb{R}^{2d}$  with nonempty intersection, it follows that  $K_{b,U_1} = K_{b,U_1 \cup U_2} = K_{b,U_2}$  in  $\mathcal{D}'^*(U_1 \cap U_2)$ . Because  $\mathcal{D}'^*$  is a sheaf (see [6]),  $K_b$  can be defined as an element of  $\mathcal{D}'^*(\mathbb{R}^{2d})$  as the oscillatory integral (29).  $\square$

**Example 5.1.** Interesting such symbols  $a$  are given by  $e^{l|x|^2} P(\xi)$ , where  $l < 1$  and  $P(\xi)$  is an ultrapolynomial of class  $*$ . In this case, obviously  $a \in B_{-1}^*$ . Moreover

$$\begin{aligned} b(x, \xi) &= \frac{1}{\pi^d} e^{-|x|^2 - |\xi|^2} \mathcal{L} \left( e^{-|\cdot|^2 - |\cdot|^2} a(\cdot, \cdot) \right) (-2x, -2\xi) \\ &= \frac{1}{\pi^d} \left( \frac{\pi}{1-l} \right)^{d/2} e^{l|x|^2/(1-l)} \int_{\mathbb{R}^d} e^{-|\eta|^2} P(\xi - \eta) d\eta \end{aligned}$$

In the  $(M_p)$  case, there exist  $m, C_1 > 0$  such that  $|P(\xi - \eta)| \leq C_1 e^{M(m|\xi|)} e^{M(m|\eta|)}$ , resp. in the  $\{M_p\}$  case, there exist  $C_1 > 0$  and  $(k_p) \in \mathfrak{R}$ , such that  $|P(\xi - \eta)| \leq C_1 e^{N_{k_p}(|\xi|)} e^{N_{k_p}(|\eta|)}$  (in the  $(M_p)$  case this estimate follows from proposition 4.5 of [6], in the  $\{M_p\}$  case the estimate easily follows by combining proposition 4.5 of [6] and lemma 3.4 of [8]). Hence,  $b$  satisfies the conditions in the above theorem and  $b^w$  can be defined as the operator corresponding to the kernel  $K_b$  defined as the oscillatory integral (29).

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